Pressure Dependent Viscosity Model for Granular Media Obtained from Compressible Navier-Stokes Equations

August 24, 2015

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Abstract

The aim of this article is to justify mathematically, in the two-dimensional periodic setting, a generalization of a two-phase model with pressure dependent viscosity first proposed by A. Lefebvre–Lepot and B. Maury in *Adv. Math. Sci. Appl.* (2011) to describe a one–dimensional system of aligned spheres interacting through lubrication forces. This model involves an adhesion potential apparent only on the congested domain, which keeps track of history of the flow. The solutions are constructed (through a singular limit) from a compressible Navier-Stokes system with viscosity and pressure both singular close to a maximal volume fraction. Interestingly, this study can be seen as the first mathematical connection between models of granular flows and models of suspension flows. As a by-product of this result, we also obtain global existence of weak solutions for a system of incompressible Navier-Stokes equations with pressure dependent viscosity, the adhesion potential playing a crucial role in this result.

MSC2010 : 35Q30, 76N10, 76T20, 76T25.

Keywords : Compressible Navier-Stokes Equations, Suspension Flows, Granular Flows, Free Boundary Problem, Pressure Dependent Viscosity, Global Weak Solutions.

1   Introduction

In most of mathematical studies on the Navier-Stokes equations it is assumed that the viscosity is either constant or depends on the temperature and on the density in the compressible case. However, it is known that viscosities in real fluids, even incompressible ones, may vary not only with the temperature but also with the pressure. In his seminal paper [35] on fluid motion, Stokes already mentionned the possibility that the viscosity of a fluid may depend on the pressure. As explained by J. Hron, J. Málek and K.R. Rajagopal in [18] such dependence is for instance relevant for fluids at high pressures and flows involving lubricants. Another

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example of pressure-dependent viscosities is provided by the theory of dense granular flows. The similarities shared by these types of flows with non-newtonian flows such as Bingham fluids, yield P. JOP, Y. FORTERRE and O. POULIQUEN to propose in [19] a constitutive law based on a dimensionless number, the "inertial number" I. In their model, called "the $\mu(I)$-rheology", the volume fraction is linked to the inertial number $\Phi = \Phi(I)$ and there exists a relation between the pressure P, the shear stress τ and the shear rate $\mathrm{D}(u)$, u being the velocity of the fluid

$$\tau = \mu(I)P \quad \text{with} \quad I = \frac{2d}{(P/\varrho_s)^{0.5}}|\,\mathrm{D}(u)|,$$

d and ϱ_s being respectively the particle diameter and the particle density.

An important feature of granular flows is the existence of a maximal volume fraction Φ^*, a constant approximatively equal to 0.64 which corresponds to the random close packing. Taking account of such a congestion constraint leads to propose a model which can describe both the free/compressible regions where $\Phi < \Phi^*$ and the congested/incompressible regions corresponding to $\Phi = \Phi^*$. The problem can thus be seen as a free boundary problem between the two subdomains. In dimension one, A. LEFEBVRE–LEPOT and B. MAURY in [22] proposed the following system which takes account of the previous constraints

$$\begin{cases} \partial_t \Phi + \partial_x(\Phi u) = 0 \\ 0 \leq \Phi \leq \Phi^* \\ \partial_t(\Phi u) + \partial_x(\Phi u^2) + \partial_x p = 0 \\ \partial_t P_a + \partial_x(P_a u) = -p \\ P_a \leq 0, \qquad P_a(\Phi^* - \Phi) = 0 \end{cases} \tag{1}$$

The idea of the variable P_a comes from [25] where it is seen as the adhesion potential of a single particle against a wall and measures in a certain sense smallness of the wall-particle distance.

This article proposes to investigate a certain generalization of the previous system in the two–dimensional case

$$\begin{cases} \partial_t \Phi + \mathrm{div}\,(\Phi\,u) = 0 & \text{(2a)} \\ 0 \leq \Phi \leq \Phi^* & \text{(2b)} \\ \partial_t(\Phi u) + \mathrm{div}\,(\Phi u \otimes u) + \nabla\Pi - \nabla\Lambda - 2\mathrm{div}\,\big((\Phi + \Pi)\,\mathrm{D}(u)\big) + r\Phi|u|u = 0 & \text{(2c)} \\ \partial_t\Pi + \mathrm{div}\,(\Pi u) = -\dfrac{\Lambda}{2} & \text{(2d)} \\ \Pi \geq 0, \qquad \Pi(\Phi^* - \Phi) = 0 & \text{(2e)} \end{cases}$$

Note that compared to (1), we have three extra terms namely $\nabla\Pi$, $-2\mathrm{div}((\Phi + \Pi)D(u))$ and $r\Phi|u|u$. The first two terms encode respectively the effect of some pressure law in the suspension model and the effect of the shear viscosity coming from the multi-dimensional setting whereas the last term, $r\Phi|u|u$, represents the friction.

Following the ideas previously developed in [11] and [31], we approximate this system by a compressible Navier-Stokes system with singular (close to Φ^*) pressure π_ε. We also consider, and this is new compared to [11] and [31], volume fraction dependent viscosities μ_ε, λ_ε singular

close to Φ^*

$$\begin{cases} \partial_t \Phi_\varepsilon + \mathrm{div}\,(\Phi_\varepsilon u_\varepsilon) = 0 & \text{(3a)} \\ \partial_t(\Phi_\varepsilon u_\varepsilon) + \mathrm{div}\,(\Phi_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla \pi_\varepsilon(\Phi_\varepsilon) + r\Phi_\varepsilon |u_\varepsilon| u_\varepsilon & \\ \qquad\qquad - \nabla(\lambda_\varepsilon(\Phi_\varepsilon)\mathrm{div}\,u_\varepsilon) - 2\mathrm{div}\,(\mu_\varepsilon(\Phi_\varepsilon)\,\mathrm{D}\,(u_\varepsilon)) = 0 & \text{(3b)} \end{cases}$$

It is then expected that $\pi_\varepsilon(\Phi_\varepsilon)$ converges towards Π, $\mu_\varepsilon(\Phi_\varepsilon)$ towards $\Phi + \Pi$ and $\lambda_\varepsilon(\Phi_\varepsilon)\mathrm{div}\,u_\varepsilon$ towards Λ. As explained in [31], the singular pressure π_ε is not only useful for numerics, since it ensures automatically the constraint $\Phi_\varepsilon \leq \Phi^*$, but is also relevant from a physical point of view. It is indeed well-known in the kinetic theory of dense gases (see [14]) that the interaction between the molecules becomes strongly repulsive at very short distance. This effect comes essentially from an electrostatic force due to the fact that the electron clouds of different atoms or molecules cannot mix together. Several empirical formula have been proposed to describe this force (see for instance the general book [15], or the famous paper [13] for a particular potential called the Carnahan-Starling potential), the common point of all of them is to consider singular potentials going to infinity faster than all the other forces involved in the model. Coming back to the theory of granular media, such repulsive pressures are also taken into account in the description of granular gases. In the gas regime, B. ANDREOTTI, Y. FORTERRE and O. POULIQUEN describe in their book [1] the kinetic theory that has been developed based on the principles of Boltzmann and Enskog. In particular some models involve the Carnahan-Starling potential.

In the liquid regime of granular flows, one needs then to take account not only of the singular pressure but also of a singular viscosity. Studying experimentally the case of suspensions (mixtures of fluid and grain in a dilute regime), one can define an effective viscosity of the mixture which is shown to vary with the volume fraction Φ and which is expected to diverge close to the maximal volume fraction Φ^* (see the books previously mentionned : [1] and [15]). As for the kinetic theory, only empirical laws are avalaible. From a mathematical point of view, to the author's knowledge, there are few mathematical studies on fluid models with singular viscosities. However, one can cite the interesting paper [22], where A. LEFEBVRE-LEPOT and B. MAURY study a simple model in one space dimension of aligned spheres interacting through lubrication forces. From solutions of the discrete model, they construct a micro-macro operator and prove the weak convergence of the solutions towards global weak solutions of the continuous Stokes system ($\Phi^* = 1$)

$$\begin{cases} -\partial_x\left(\dfrac{1}{1-\Phi}\partial_x u\right) = \Phi f \\ \partial_t \Phi + \partial_x(\Phi u) = 0. \end{cases}$$

To the author's knowledge, this seems to be the first mathematical justification of the presence of a singular viscosity in a coupled system.

For singular (close to Φ^*) viscosities and asymptotic two–phase description, it seems that nothing is known concerning mathematical justification. The problem has been envisaged by A. LEFEBVRE-LEPOT and B. MAURY in [22]. At the end of this paper they suggest that the singular system

$$\begin{cases} \partial_t \Phi + \partial_x(\Phi u) = 0 \\ \partial_t(\Phi u) + \partial_x(\Phi u^2) - \partial_x\left(\dfrac{\varepsilon}{1-\Phi}\partial_x u\right) = \Phi f \end{cases}$$

could converge as $\varepsilon \to 0$ towards the hybrid system

$$
\begin{cases}
\partial_t \Phi + \partial_x(\Phi u) = 0 \\
\partial_t(\Phi u) + \partial_x(\Phi u^2) + \partial_x p = f \\
\partial_t P_a + \partial_x(P_a u) = -p \\
P_a \leq 0, \quad \Phi \leq 1, \quad P_a(1 - \Phi) = 0
\end{cases}
\tag{4}
$$

previously presented. Nevertheless, the singular limit passage $\varepsilon \to 0$ towards the hybrid Navier-Stokes system is not rigourously proven. Note that in the one-dimensional setting div and ∇ are the same and also that they do not consider pressure in the momentum equation. This is the main difference between our mathematically justified asymptotic system (2) and the proposed limit system (4).

In this paper we want to take account of convection and of a singular pressure, a natural question is then to know if we have to impose a relationship between the singular viscosities and the singular pressure. An interesting remark for our study which can be found in [1] is basically the following: if one wants to describe within the same framework suspensions and immersed granular media (described by the $\mu(I)$-rheology introduced before), one has to ensure the compatibilty of the two formulations by imposing the same divergence in the viscosity and the pressure close to Φ^*. This is the approach followed in this paper where the shear viscosity μ_ε and the pressure π_ε increase exponentially close to Φ^*

$$
\mu_\varepsilon(\Phi) = \frac{\Phi}{\varepsilon} \left(\exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi/\Phi^*} \right) - 1 \right) + \Phi,
$$

$$
\lambda_\varepsilon(\Phi) = 2\varepsilon^a \frac{\Phi^2}{\Phi^*\left(1 - \dfrac{\Phi}{\Phi^*}\right)^2} \exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi/\Phi^*} \right)
$$

$$
\pi_\varepsilon(\Phi) = \frac{\Phi}{\varepsilon} \left(\frac{\Phi}{\Phi^*} \right)^\gamma \left(\exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi/\Phi^*} \right) - 1 \right)
$$

with Φ^* a fixed constant, $a > 1$ and $\gamma \geq 1$.

The mathematical justification of the limit passage $\varepsilon \to 0$ from a compressible model of type (3) with a singular pressure π_ε towards a two-phase model of type (2) has been the subject of two recent articles, [11] and [31] respectively in the one-dimensional setting and in the three-dimensional setting with an additional heterogeneity in the congestion constraint. Nevertheless, these papers concern only constant viscosities and cannot therefore cover the case of dense suspensions for which we have seen that the viscosities depend on the volume fraction. To answer this question we need to carefully study the compatibility between the estimates derived from compressible Navier-Stokes equations with volume fraction-dependent viscosities and the singular limit passage $\varepsilon \to 0$.

More precisely, considering "degenerate viscosities" (meaning that viscosity $\mu(\Phi)$ vanishes on the vacuum, $\Phi = 0$), one cannot deduce from the energy estimate a control on the gradient of the velocity contrary to the constant case. To deal with this difficulty, D. BRESCH and B. DESJARDINS proposed in [7], [8] a new entropy for the compressible Navier Stokes system

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with degenerate viscosities. First for the shallow water viscosity $\mu(\Phi) = \mu^0\Phi$, $\lambda(\Phi) = 0$ (see also [6]), then for more general viscosities $\mu(\Phi)$, $\lambda(\Phi)$ satisfying the algebraic relation $\lambda(\Phi) = 2(\mu'(\Phi)\Phi - \mu(\Phi))$. The idea is to introduce the effective velocity $w = u + 2\nabla\varphi(\Phi)$ where φ is linked to the viscosity by the relation

$$\varphi'(\Phi) = \frac{\mu'(\Phi)}{\Phi}$$

and to derive the energy associated to this velocity. In [9], D. BRESCH, B. DESJARDINS and D. GÉRARD-VARET proved the stability of the solutions for the compressible Navier-Stokes equations with additional terms such as drag terms or a singular (close to 0) pressure. With no extra terms, the stability result is given by a new estimate derived by A. MELLET and A. VASSEUR in [28]. This estimate provides then the extra-integrability on $\sqrt{\Phi}u$ necessary to pass to the limit in the convective term of the momentum equation $\Phi u \otimes u$.

In the shallow water case $\mu(\Phi) = \Phi$, $\lambda(\Phi) = 0$ with friction or cold pressure, D. BRESCH and B. DESJARDINS gave some hints in [8] to built a sequence of approximate solutions compatible with the BD entropy and E. ZATORSKA in [37], [38] gave the complete proof of existence of weak solutions. Note that for the shallow water sytem with no drag terms nor cold pressure it is also possible to construct global weak solutions. The idea developed by A. VASSEUR and C. YU in [36] is to consider the system with drag terms and a quantum potential, then to construct a smooth multiplier allowing to get the Mellet-Vasseur estimate which does not depend on the drag. It is then possible to let the drag term go to 0 in the equations to recover weak solutions of the classical compressible Navier-Stokes system.

For the general Navier-Stokes system with the algebraic relation $\lambda(\Phi) = 2(\mu'(\Phi)\Phi - \mu(\Phi))$ construction of approximate solutions satisfying the energy and the BD entropy is not easy even with the hints given in [8]. In [10], D. BRESCH, B. DESJARDINS and E. ZATORSKA propose a new concept of global weak solutions called κ-entropy solutions which is based on a generalization of the BD entropy. Considering the energy associated to the velocity $w = u + 2\kappa\nabla\varphi$, where $\kappa \in (0,1)$ is a fixed parameter, they derive the κ-entropy estimate. This notion of weak solution is weaker than the previous based on the energy and the BD entropy in the sense that a global weak solution of the compressible Navier-Stokes equations which satisfies the energy and the BD entropy is also a κ-entropy solution for all $0 \leq \kappa \leq 1$. Yet, the opposite claim may not be true.

In the framework of degenerate viscosities, to the knowledge of the author, the only work justifying the limit passage from the compressible system towards the two-phase model concerns the shallow water equations ($\mu(\Phi) = \Phi$, $\lambda(\Phi) = 0$) with capillarity and a power law pressure $a\Phi^\gamma$ with $\gamma \to +\infty$ (see [21]). Moreover the authors need to multiply the weak formulation of the momentum equation by Φ to deal with the possible vacuum states $\Phi = 0$.

The objective of the present paper is to address first the global existence of weak solutions in the two-dimensionnal periodic setting to the suspension model (3) with the singular pressure π_ε and singular viscosities $\mu_\varepsilon, \lambda_\varepsilon$ introduced before and an additional friction term necessary to ensure the limit passage in the convective term $\Phi_\varepsilon u_\varepsilon \otimes u_\varepsilon$, as explained previously. In dimension 2, imposing the same divergence on π_ε and μ_ε we prove the global existence of κ-entropy solutions satisfying the constraint $0 \leq \Phi_\varepsilon \leq \Phi^*$.

The second part of the article consists of the justification of the singular limit $\varepsilon \to 0$ towards the two-phase system (2) modelling a granular media. Compared to the previous work with constant viscosities [31], it is interesting to note that the singularity of the viscosity simplifies some compactness arguments and brings more regularity on the limit pressure. Indeed, the κ-entropy gives then a control in dimension 2 of all the powers of μ_ε and in particular, since we have chosen the same divergence on the pressure and the viscosity, this implies a control of the singular pressure π_ε with no need of additional estimate.

All this study strongly relies on the uniform $L^\infty(0,T;L^p(\Omega))$ controls, $p \in [1,+\infty)$ of the singular (close to Φ^*) coefficients derived from the κ-entropy estimate due to the fact that the space dimension is equal to 2.

As a corollary of our result, if $\Pi^0 > 0$, taking as initial volume fraction $\Phi^0 = \Phi^*$ and approximating system (2) by (3) with an appropriate initial datum $(u_\varepsilon^0, \Pi^0(\Phi_\varepsilon^0), \Phi_\varepsilon^0)$ converging to (u^0, Π^0, Φ^*), we obtain weak solutions to the fully incompressible system

$$
\begin{cases}
\operatorname{div} u = 0 & \text{(5a)} \\[2mm]
\partial_t u + u \cdot \nabla u + \dfrac{1}{\Phi^*} \nabla (\Pi - \Lambda) - 2\operatorname{div} \left(\left(\dfrac{\Pi}{\Phi^*} + 1 \right) \mathrm{D}(u) \right) + r|u|u = 0 & \text{(5b)} \\[2mm]
\Pi \geq 0, \quad \partial_t \Pi + \operatorname{div}(u\Pi) = -\dfrac{\Lambda}{2} & \text{(5c)}
\end{cases}
$$

There have been few mathematical studies concerning incompressible flows with general pressure dependent viscosities. Most of the works, see for instance the interesting review paper [27] by J. MÁLEK and K.R. RAJAGOPAL or the article [12], deal with a viscosity depending on both the pressure and the shear rate

$$
\mu = \mu(p, |\mathrm{D}(u)|^2)
$$

with an implicit relation between the Cauchy stress tensor and the shear rate $\mathrm{D}(u)$. It seems to be no global existence theory for purely pressure dependent viscosity. In [32], M. RENARDY confirms the physical relevance of a linear dependence of viscosity with respect to the pressure; indeed he proves that pressure driven parallel flow exists only if the viscosity is a linear function of the pressure. But M. RENARDY can establish (c.f. [32]) existence and uniqueness of solutions only under a restriction on the velocity field : the eigenvalues of $\mathrm{D}(u)$ have to be strictly less than

$$
\frac{1}{\lim_{p\to\infty} \mu'(p)}.
$$

Later, F. GAZZOLA showed in [17] a local existence result without the previous restrictions, but for small data and assuming an exponential dependance of the viscosity with respect to the pressure .

It seems then that there is no equivalent of our result in the literature on incompressible flows with pressure dependent viscosity. We obtain a global existence result of weak solutions with no restriction on the initial data nor an unrealistic relationship between the viscosity and the pressure. Although the ratio μ/p tends in our study to 1 as $p \to +\infty$ and not to $+\infty$ as suggested in [27], our constraint is consistent with the arguments developed for the theory of granular flows in [1]. Note the important role played by the adhesion potential Π

in our mathematical results since it is this potential, via the equation (5c), which provides the additional estimate necessary to prove the stability of the solutions of the incompressible system (5).

2 The suspension and the two–phase granular systems

As mentionned in the introduction our study restricts in dimension 2. To ensure the sufficient controls and the compactness of all singular quantities, in all the paper Ω will be the periodic domain $\mathbb{T}^2$. We consider the two-phase granular system

$$
\begin{cases}
\partial_t \Phi + \operatorname{div} (\Phi\, u) = 0 & \text{(6a)} \\[2mm]
0 \leq \Phi \leq \Phi^* & \text{(6b)} \\[2mm]
\partial_t \Pi + \operatorname{div} (\Pi u) = -\dfrac{\Lambda}{2} & \text{(6c)} \\[2mm]
\partial_t(\Phi u) + \operatorname{div} (\Phi u \otimes u) + \nabla\Pi - \nabla\Lambda - 2\operatorname{div}\left((\Pi + \Phi)\,\mathrm{D}\,(u)\right) + r\Phi|u|u = 0 & \text{(6d)} \\[2mm]
\Phi\,\Pi = \Phi^*\,\Pi \geq 0 & \text{(6e)}
\end{cases}
$$

with Φ^* a positive constant which represents the maximal volume fraction and $r > 0$ a small coefficient which will be determined in the proof. We supplement the system by initial conditions

$$
\Phi_{|t=0} = \Phi^0, \quad (\Phi u)_{|t=0} = m^0
$$

$$
\Pi^0 \in L^\infty(\Omega) \cap W^{1,2}(\Omega) \tag{7}
$$

with

$$
0 \leq \Phi^0 \leq \Phi^* \tag{8}
$$

$$
\frac{|m^0|^2}{\Phi^0} = 0 \quad \text{a.e. on} \quad \{x \in \Omega\,;\, \Phi^0(x) = 0\}, \quad \frac{|m^0|^2}{\Phi^0} \in L^1(\Omega) \tag{9}
$$

Remark: As said above, the parameter r is small, precisely it is taken sufficiently small to ensure the compatibility of the system with the κ-entropy inequality as it will be explained in Section 3.1.

2.1 A model for suspension flows based on singular Compressible Navier-Stokes equations

We approximate the previous two-phase system by mean of a singular pertubation, we will call this perturbed system the "suspension model"

$$
\begin{cases}
\partial_t \Phi_\varepsilon + \operatorname{div} (\Phi_\varepsilon u_\varepsilon) = 0, & \text{(10a)} \\[2mm]
\partial_t(\Phi_\varepsilon u_\varepsilon) + \operatorname{div} (\Phi_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla\pi_\varepsilon(\Phi_\varepsilon) + r\Phi_\varepsilon|u_\varepsilon|u_\varepsilon & \\[2mm]
\qquad\qquad -2\operatorname{div}\left(\mu_\varepsilon(\Phi_\varepsilon)\,\mathrm{D}\,(u_\varepsilon)\right) - \nabla(\lambda_\varepsilon(\Phi_\varepsilon)\operatorname{div} u_\varepsilon) = 0 & \text{(10b)}
\end{cases}
$$

The viscosities are defined by

$$\mu_\varepsilon(\Phi) = \mu_\varepsilon^1(\Phi) + \Phi = \begin{cases} \dfrac{1}{\varepsilon}\Phi\left(\exp\left(\dfrac{\varepsilon^{1+a}}{1-\dfrac{\Phi}{\Phi^*}}\right) - 1\right) + \Phi & \text{if } \dfrac{\Phi}{\Phi^*} < 1 \\[2em] +\infty & \text{if } \dfrac{\Phi}{\Phi^*} \geq 1 \end{cases} \tag{11}$$

with $a > 1$ and the algebraic relation introduced by BRESCH and DESJARDINS in [7]

$$\lambda_\varepsilon(\Phi) = 2(\mu_\varepsilon'(\Phi)\Phi - \mu_\varepsilon(\Phi)). \tag{12}$$

The singular pressure is related to these viscosities and is defined by

$$\pi_\varepsilon(\Phi) = \left(\dfrac{\Phi}{\Phi^*}\right)^\gamma \mu_\varepsilon^1(\Phi) = \begin{cases} \dfrac{\Phi}{\varepsilon}\left(\dfrac{\Phi}{\Phi^*}\right)^\gamma\left(\exp\left(\dfrac{\varepsilon^{1+a}}{1-\dfrac{\Phi}{\Phi^*}}\right) - 1\right) & \text{if } \dfrac{\Phi}{\Phi^*} < 1 \\[2em] +\infty & \text{if } \dfrac{\Phi}{\Phi^*} \geq 1 \end{cases} \tag{13}$$

with

$$\gamma \geq 0. \tag{14}$$

Remark: Formally we observe that $a > 0$ ensures the convergence to 0 as $\varepsilon \to 0$ of the singular terms μ_ε^1, λ_ε and π_ε on the set $\{\Phi < \Phi^*\}$ but we will see in the proof of Lemma 2 that we need $a > 1$ to guarantee the convergence of $(1 - \Phi_\varepsilon/\Phi^*)\pi_\varepsilon(\Phi_\varepsilon)$ towards 0 and obtain (6e).

The system (10a)–(10b) is supplemented by initial data $(\Phi_\varepsilon^0, m_\varepsilon^0)$

$$0 \leq \Phi_\varepsilon^0 < \Phi^* \tag{15}$$

$$\dfrac{|m_\varepsilon^0|^2}{\Phi_\varepsilon^0} = 0 \quad \text{a.e. on} \quad \{x \in \Omega\,;\Phi_\varepsilon^0(x) = 0\}, \quad \dfrac{|m_\varepsilon^0|^2}{\Phi_\varepsilon^0} \in L^1(\Omega) \tag{16}$$

$$\dfrac{\nabla\mu_\varepsilon(\Phi_\varepsilon^0)}{\sqrt{\Phi_\varepsilon^0}} \in L^2(\Omega), \quad \mu_\varepsilon(\Phi_\varepsilon^0) \in L^1(\Omega) \tag{17}$$

$$\Phi_\varepsilon^0 e_\varepsilon(\Phi_\varepsilon^0) \in L^1(\Omega) \tag{18}$$

where e_ε is such that

$$e_\varepsilon'(\Phi) = \dfrac{\pi_\varepsilon(\Phi)}{\Phi^2},$$

and where all the bounds are uniform with respect to ε.

Let us now introduce the notion of weak solution for system (10a)–(10b).

Definition 1 (κ-entropy solutions of (10)) *Let $T > 0$ and $\kappa \in (0,1)$, $(\Phi_\varepsilon, u_\varepsilon)$ is called a κ-entropy solution to system (10a)–(10b), under the initial conditions (15)–(18) if it satisfies*

- meas$\{(t,x)\ :\ \Phi_\varepsilon(t,x) \geq 1\} = 0;$

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- *the mass equation in the weak sense*

$$-\int_0^T \int_\Omega \Phi_\varepsilon \partial_t \xi - \int_0^T \int_\Omega \Phi_\varepsilon u_\varepsilon \cdot \nabla \xi = \int_\Omega \Phi_\varepsilon^0 \xi(0) \qquad \forall \xi \in \mathcal{D}([0,T) \times \Omega); \qquad (19)$$

- *the momentum equation in the weak sense*

$$-\int_0^T \int_\Omega \Phi_\varepsilon u_\varepsilon \cdot \partial_t \zeta - \int_0^T \int_\Omega (\Phi_\varepsilon u_\varepsilon \otimes u_\varepsilon) : \nabla \zeta - \int_0^T \int_\Omega \pi_\varepsilon(\Phi_\varepsilon) \mathrm{div}\, \zeta$$
$$+ r \int_0^T \int_\Omega \Phi_\varepsilon |u_\varepsilon| u_\varepsilon \cdot \zeta + 2 \int_0^T \int_\Omega \mu_\varepsilon(\Phi_\varepsilon) \, \mathrm{D}\,(u_\varepsilon) : \nabla \zeta$$
$$+ \int_0^T \int_\Omega \lambda_\varepsilon(\Phi_\varepsilon) \mathrm{div}\, u_\varepsilon \mathrm{div}\, \zeta = \int_\Omega m_\varepsilon^0 \cdot \zeta(0) \qquad (20)$$

for all $\zeta \in \left(\mathcal{D}([0,T) \times \Omega)\right)^2$.

- *the κ-entropy inequality*

$$\sup_{t \in [0,T]} \left[\int_\Omega \Phi_\varepsilon \left(\frac{|u_\varepsilon + 2\kappa \nabla \varphi_\varepsilon(\Phi_\varepsilon)|^2}{2} + \kappa(1-\kappa) \frac{|2 \nabla \varphi_\varepsilon(\Phi_\varepsilon)|^2}{2} \right) + \Phi_\varepsilon e_\varepsilon(\Phi_\varepsilon) \right]$$
$$+ r \sup_{t \in [0,T]} \int_\Omega \mu_\varepsilon(\Phi_\varepsilon) + r \int_0^T \int_\Omega \Phi_\varepsilon |u_\varepsilon|^3$$
$$+ \kappa \int_0^T \int_\Omega \mu_\varepsilon(\Phi_\varepsilon) |\mathrm{A}\,(u_\varepsilon)|^2 + 2\kappa \int_0^T \int_\Omega \mu_\varepsilon'(\Phi_\varepsilon) \frac{\pi_\varepsilon'(\Phi_\varepsilon)}{\Phi_\varepsilon} |\nabla \Phi_\varepsilon|^2$$
$$+ (1-\kappa) \int_0^T \left[\int_\Omega \mu_\varepsilon(\Phi_\varepsilon) |\mathrm{D}\,(u_\varepsilon)|^2 + \int_\Omega \left(\mu_\varepsilon'(\Phi_\varepsilon)\Phi_\varepsilon - \mu_\varepsilon(\Phi_\varepsilon) \right) |\mathrm{div}\, u_\varepsilon|^2 \right]$$
$$\leq C(r) \qquad (21)$$

where φ_ε *is such that*

$$\varphi_\varepsilon'(\Phi) = \frac{\mu_\varepsilon'(\Phi)}{\Phi},$$

and $C(r)$ *is a constant which depends only on* r *and on the initial data* $(\Phi_\varepsilon^0, m_\varepsilon^0)$.

Remark: $\mathrm{D}\,(u)$ and $\mathrm{A}\,(u)$ are respectively the symmetric and the antisymmetric parts of the gradient defined by

$$\mathrm{D}\,(u) = \frac{\nabla u + \nabla^t u}{2}, \qquad \mathrm{A}\,(u) = \frac{\nabla u - \nabla^t u}{2}.$$

Remark: To prove the stability of the solutions the integrals of the diffusion terms

$$2 \int_0^T \int_\Omega \mu_\varepsilon(\Phi_\varepsilon) \, \mathrm{D}\,(u_\varepsilon) : \nabla \zeta + \int_0^T \int_\Omega \lambda_\varepsilon(\Phi_\varepsilon) \mathrm{div}\,(u_\varepsilon) \mathrm{div}\, \zeta$$

should be understood in the following sense

$$2 \int_0^T \int_\Omega \mu_\varepsilon(\Phi_\varepsilon) \, \mathrm{D}\,(u_\varepsilon) : \nabla \zeta = -\int_0^T \int_\Omega \sqrt{\Phi_\varepsilon} u_\varepsilon^j \left(\frac{\partial_i \mu_\varepsilon(\Phi_\varepsilon)}{\sqrt{\Phi_\varepsilon}} \partial_i \zeta^j + \frac{\mu_\varepsilon(\Phi_\varepsilon)}{\sqrt{\Phi_\varepsilon}} \partial_{ii}^2 \zeta^j \right)$$
$$-\int_0^T \int_\Omega \sqrt{\Phi_\varepsilon} u_\varepsilon^i \left(\frac{\partial_j \mu_\varepsilon(\Phi_\varepsilon)}{\sqrt{\Phi_\varepsilon}} \partial_i \zeta^j + \frac{\mu_\varepsilon(\Phi_\varepsilon)}{\sqrt{\Phi_\varepsilon}} \partial_{ij}^2 \zeta^j \right) \qquad (22)$$

and

$$\int_0^T \int_\Omega \lambda_\varepsilon(\Phi_\varepsilon) \mathrm{div}\, u_\varepsilon \mathrm{div}\, \zeta = \int_0^T \int_\Omega \frac{\lambda_\varepsilon(\Phi_\varepsilon)}{\sqrt{\mu_\varepsilon(\Phi_\varepsilon)}} \sqrt{\mu_\varepsilon(\Phi_\varepsilon)} \mathrm{div}\, u_\varepsilon \mathrm{div}\, \zeta. \tag{23}$$

Remark: Compared to the work of BRESCH, DESJARDINS, ZATORSKA [10], we have the additional integral

$$r \sup_{t \in [0,T]} \int_\Omega \mu_\varepsilon(\Phi_\varepsilon)$$

which allows us to deduce the control of $\mu_\varepsilon(\Phi_\varepsilon)$ in $L^\infty(0,T;L^p(\Omega))$ for $p \in [1,+\infty)$ if we combine it with the control of $\nabla \mu_\varepsilon(\Phi_\varepsilon)$ in $L^\infty(0,T;L^q(\Omega))$ for all $q \in [1,2)$. Note that this control strongly relies on the presence of the drag term in the equations.

Remark: It would be also possible in the analysis to consider the more simpler case of pressure and viscosities singular close to Φ^* as power laws, namely

$$\pi_\varepsilon(\Phi) = \frac{\varepsilon\Phi}{\left(1 - \dfrac{\Phi}{\Phi^*}\right)^\beta}$$

as proposed in [22], keeping the relationship between the coefficients μ_ε, π_ε and λ_ε. The existence of weak solutions when ε is fixed works exactly in the same way. Modifying slightly the arguments, one can also prove the limit passage $\varepsilon \to 0$ towards the same hybrid model satisfying $(\Phi^* - \Phi)\Pi = 0$.

2.2 Main results

Under the conditions previously stated, we are able to build global weak solutions of the system (10a)–(10b).

Theorem 1 (Existence for the suspension model) *Let $T > 0$, $\varepsilon > 0$ and $(\Phi_\varepsilon^0, m_\varepsilon^0)$ an initial data satisfying (15)–(18). There exists $r > 0$, which depends on T, such that there exists a κ-entropy solution $(\Phi_\varepsilon, u_\varepsilon)$ to the suspension model (10a)–(10b) in the sense of Definition 1.*

Thanks to the previous existence result we can address now the question of the singular limit passage $\varepsilon \to 0$ towards the two–phase system.

Theorem 2 (Existence for the two–phase system) *Let $T > 0$, (Φ^0, m^0, Π^0) and $(\Phi_\varepsilon^0, m_\varepsilon^0)$ satisfy respectively (7)–(9) and (15)–(18). We assume that $\Phi_\varepsilon^0 \to \Phi^0$ in $L^p(\Omega)$ for all $p \in [1,+\infty)$, $m_\varepsilon^0/\sqrt{\Phi_\varepsilon^0} \to m^0/\sqrt{\Phi^0}$ in $L^2(\Omega)$ as $\varepsilon \to 0$. Let r such that we can apply Theorem 1. Then there exists a subsequence $(\Phi_\varepsilon, u_\varepsilon, \pi_\varepsilon(\Phi_\varepsilon), \lambda_\varepsilon(\Phi_\varepsilon)\mathrm{div}\, u_\varepsilon)$ converging to (Φ, u, Π, Λ)*

solution of

$$
\begin{cases}
-\int_0^T \int_\Omega \Phi \partial_t \xi - \int_0^T \int_\Omega \Phi u \cdot \nabla \xi = \int_\Omega \Phi^0 \xi(0) & \text{(24a)} \\[2ex]
0 \le \Phi \le \Phi^* \qquad a.e. \ in \quad (0,T) \times \Omega & \text{(24b)} \\[2ex]
-\int_0^T \int_\Omega \Pi \partial_t \xi - \int_0^T \int_\Omega \Pi \, u \cdot \nabla \xi + \langle \frac{\Lambda}{2}, \xi \rangle = \int_\Omega \Pi^0 \xi(0) & \text{(24c)} \\[2ex]
-\int_0^T \int_\Omega \Phi u \cdot \partial_t \zeta - \int_0^T \int_\Omega (\Phi u \otimes u) : \nabla \zeta - \int_0^T \int_\Omega \Pi \operatorname{div} \zeta + r \int_0^T \int_\Omega \Phi |u| u \cdot \zeta \\[2ex]
\qquad + \langle \Lambda, \operatorname{div} \zeta \rangle + 2 \langle \, (\Pi + \Phi) \operatorname{D}(u), \nabla \zeta \, \rangle = \int_\Omega m^0 \cdot \zeta(0) & \text{(24d)} \\[2ex]
\Phi \, \Pi = \Phi^* \, \Pi \ge 0 \qquad a.e. \ in \quad (0,T) \times \Omega & \text{(24e)}
\end{cases}
$$

for all $\xi \in \mathcal{D}([0,T] \times \Omega)$, $\zeta \in (\mathcal{D}([0,T] \times \Omega))^2$ and where the terms $\langle \frac{\Lambda}{2}, \xi \rangle$ and $\langle \Lambda, \operatorname{div} \zeta \rangle$ have to be understood as the duality pairing between the distribution Λ and the test functions ξ, $\operatorname{div} \zeta$. Moreover, the limit has the following regularity

$$\Phi \in \mathcal{C}([0,T]; L^p(\Omega)) \cap L^\infty(0,T; W^{1,2}(\Omega)), \quad for \ all \quad 1 \le p < +\infty,$$

$$\Pi \in L^\infty(0,T; W^{1,2}(\Omega)),$$

$$\sqrt{\Pi + \Phi} \, \operatorname{D}(u) \in L^2((0,T) \times \Omega),$$

$$\sqrt{\Phi} u \in L^\infty(0,T; L^2(\Omega)).$$

$$\Lambda \in W^{-1,\infty}(0,T; L^p(\Omega)) + L^\infty(0,T; W^{-1,q}(\Omega)) \quad for \ all \quad p \in [1,+\infty), q \in [1,2).$$

Remark: We observe that we get much more regularity on the limit pressure Π than in the constant viscosities case [31]. As we will see in the proof, this is a consequence of the κ-entropy and the relationship satisfied by μ_ε and π_ε. In particular this regularity gives a sense to the product

$$
\begin{aligned}
\langle \, \Pi \operatorname{D}(u), \nabla \zeta \, \rangle = & -\int_0^T \int_\Omega \sqrt{\Phi} u^j \left(\frac{\partial_i \Pi}{\sqrt{\Phi}} \partial_i \zeta^j + \frac{\Pi}{\sqrt{\Phi}} \partial_{ii}^2 \zeta^j \right) \\
& -\int_0^T \int_\Omega \sqrt{\Phi} u^i \left(\frac{\partial_j \Pi}{\sqrt{\Phi}} \partial_i \zeta^j + \frac{\Pi}{\sqrt{\Phi}} \partial_{ij}^2 \zeta^j \right)
\end{aligned}
\tag{25}
$$

The difficulty in the proof of Theorem 2, compared to the case $\varepsilon > 0$ relies on the fact that at the limit $\varepsilon = 0$ we do no have meas $\{(t,x) \ : \ \Phi(t,x) = \Phi^*\} = 0$. We then need to carefully study the control that we have on the singular coefficients taking into account the possible convergence of Φ_ε towards Φ^*.

Global existence of weak solutions to an incompressible Navier-Stokes system with pressure dependent viscosity

As in the constant viscosities case studied by P.–L. LIONS and N. MASMOUDI in [24], we prove in Section 5 the compatibility on the limit system between the constraint (24b) and the divergence free condition

$$\operatorname{div} u = 0 \quad \text{a.e. in} \quad \{\Phi = \Phi^*\}. \tag{26}$$

If initially the two–phase system is entirely congested, meaning that $\Phi^0 = \Phi^*$, $\Pi^0 > 0$ and $\operatorname{div} u^0 = 0$, then, considering the approximated singular system (10a)–(10b) with initially $\Phi^0_\varepsilon = \Phi^* \left(1 - \varepsilon^a \dfrac{\Phi^*}{\Pi^0} \right)$, the previous theorem will give us the existence of global weak solutions for the incompressible system with pressure dependent viscosity.

Theorem 3 (Existence for the incompressible system) *Let $T > 0$, (u^0, Π^0) such that $u^0 \in L^2(\Omega)$, $\operatorname{div} u^0 = 0$ and $\Pi^0 > 0$, with*

$$\Pi^0 \in L^\infty(\Omega) \cap W^{1,2}(\Omega).$$

There exists a global weak solution to the pressure dependent incompressible system for all $\xi \in \mathcal{D}([0,T) \times \Omega)$, $\zeta \in (\mathcal{D}([0,T) \times \Omega))^2$

$$
\begin{cases}
\operatorname{div} u = 0 & (27a) \\[2mm]
\displaystyle -\int_0^T \int_\Omega \Pi \partial_t \xi - \int_0^T \int_\Omega \Pi u \cdot \nabla \xi + \langle \frac{\Lambda}{2}, \xi \rangle = \int_\Omega \Pi^0 \xi(0) & (27b) \\[2mm]
\displaystyle -\int_0^T \int_\Omega u \cdot \partial_t \zeta - \int_0^T \int_\Omega (u \otimes u) : \nabla \zeta - \int_0^T \int_\Omega \frac{\Pi}{\Phi^*} \operatorname{div} \zeta + r \int_0^T \int_\Omega |u| u \cdot \zeta \\[2mm]
\displaystyle \qquad + \langle \frac{\Lambda}{\Phi^*}, \operatorname{div} \zeta \rangle + 2 \langle (\Pi/\Phi^* + 1) \, \mathrm{D}(u), \nabla \zeta \rangle = \int_\Omega u^0 \cdot \zeta(0), & (27c)
\end{cases}
$$

satisfying

$$\Pi \geq 0, \quad \Pi \in L^\infty(0,T; W^{1,2}(\Omega)),$$

$$\Lambda \in W^{-1,\infty}(0,T; L^p(\Omega)) + L^\infty(0,T; W^{-1,q}(\Omega)), \quad \forall p \in [1, +\infty), q \in [1,2),$$

$$u \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)).$$

Remark: We can recover the regularity of the potential Π directly from the system (27a)–(27c) using the BD-entropy (see for instance [6]). To simplify the explaination, we drop the drag term of the momentum equation. Taking the gradient of

$$\partial_t \Pi + \operatorname{div}(\Pi u) = -\frac{\Lambda}{2}$$

and dividing by Φ^*, we have

$$\partial_t \nabla \left(\frac{\Pi}{\Phi^*} + 1 \right) + \operatorname{div} \left(u \otimes \nabla \left(\frac{\Pi}{\Phi^*} + 1 \right) \right) + \operatorname{div} \left(\left(\frac{\Pi}{\Phi^*} + 1 \right) \nabla^t u \right) + \frac{\nabla \Lambda}{2\Phi^*} = 0$$

Then, introducing the effective velocity $w = u + 2\nabla\left(\dfrac{\Pi}{\Phi^*} + 1\right)$, w satisfies,

$$\partial_t w + \mathrm{div}\left(u \otimes w\right) + \frac{\nabla\Pi}{\Phi^*} - 2\mathrm{div}\left(\left(\frac{\Pi}{\Phi^*} + 1\right)\mathrm{A}\left(u\right)\right) + \frac{\nabla\cancel{\chi}}{\Phi^*} - \frac{\nabla\cancel{\chi}}{\cancel{\Phi}^*} = 0.$$

Finally, multiplying this last equation by w and integrating, since u is divergence free, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |w|^2 + \int_\Omega \left|\frac{\nabla\Pi}{\Phi^*}\right|^2 + 2\int_\Omega \left(\frac{\Pi}{\Phi^*} + 1\right)|\mathrm{A}\left(u\right)|^2 = 0$$

which ensures that w is in $L^\infty(0,T;W^{1,2}(\Omega))$ if initially $u^0 \in L^2(\Omega)$ and $\Pi^0 \in W^{1,2}(\Omega)$.

2.3   Sketch of the proof of Theorem 1

The proof of Theorem 1 is not a direct consequence of the theory of the κ-entropy developped in [10] since the pressure and the viscosities are singular close to Φ^*. To deal with this difficulty we first add a parameter δ in order to truncate these singular terms. Then we add an artificial pressure $\vartheta\nabla p(\Phi) = \vartheta\nabla\dfrac{\Phi^2}{2}$, $\vartheta > 0$ in order to control the gradient of the density. The approximate system reads as

$$\begin{cases}
\partial_t \Phi_{\vartheta,\delta} + \mathrm{div}\left(\Phi_{\vartheta,\delta} u_{\vartheta,\delta}\right) = 0 & (28a)\\
\partial_t(\Phi_{\vartheta,\delta} u_{\vartheta,\delta}) + \mathrm{div}\left(\Phi_{\vartheta,\delta} u_{\vartheta,\delta} \otimes u_{\vartheta,\delta}\right) + \vartheta\nabla p(\Phi_{\vartheta,\delta}) + \nabla\pi_{\varepsilon,\delta}(\Phi_{\vartheta,\delta}) + r\Phi_{\vartheta,\delta}|u_{\vartheta,\delta}|u_{\vartheta,\delta} \\
\qquad -2\mathrm{div}\left(\mu_{\varepsilon,\delta}(\Phi_{\vartheta,\delta})\,\mathrm{D}\left(u_{\vartheta,\delta}\right)\right) - \nabla(\lambda_{\varepsilon,\delta}(\Phi_{\vartheta,\delta})\mathrm{div}\left(u_{\vartheta,\delta}\right)) = 0 & (28b)
\end{cases}$$

with

$$\pi_{\varepsilon,\delta}(\Phi) = \begin{cases}
\dfrac{\Phi}{\varepsilon}\left(\dfrac{\Phi}{\Phi^*}\right)^\gamma\left(\exp\left(\dfrac{\varepsilon^{1+a}}{1 - \dfrac{\Phi}{\Phi^*}}\right) - 1\right) & \text{if}\quad \dfrac{\Phi}{\Phi^*} \le 1 - \delta \\[2em]
\dfrac{\Phi}{\varepsilon}\left(\dfrac{\Phi}{\Phi^*}\right)^\gamma\left(\exp\left(\dfrac{\varepsilon^{1+a}}{\delta}\right) - 1\right) & \text{if}\quad \dfrac{\Phi}{\Phi^*} > 1 - \delta
\end{cases} \tag{29}$$

$$\mu_{\varepsilon,\delta}(\Phi) = \begin{cases}
\dfrac{\Phi}{\varepsilon}\left(\exp\left(\dfrac{\varepsilon^{1+a}}{1 - \dfrac{\Phi}{\Phi^*}}\right) - 1\right) + \Phi & \text{if}\quad \dfrac{\Phi}{\Phi^*} \le 1 - \delta \\[2em]
\dfrac{\Phi}{\varepsilon}\left(\exp\left(\dfrac{\varepsilon^{1+a}}{\delta}\right) - 1\right) + \Phi & \text{if}\quad \dfrac{\Phi}{\Phi^*} > 1 - \delta
\end{cases} \tag{30}$$

If we denote $(\mu_{\varepsilon,\delta})'_+$ the right-derivative of $\mu_{\varepsilon,\delta}$, $\lambda_{\varepsilon,\delta}$ is related to $\mu_{\varepsilon,\delta}$ via the algebraic condition

$$\lambda_{\varepsilon,\delta}(\Phi) = 2((\mu_{\varepsilon,\delta})'_+(\Phi)\Phi - \mu_{\varepsilon,\delta}(\Phi))$$

$$= \begin{cases}
2\varepsilon^a\dfrac{\Phi^2}{\Phi^*\left(1 - \dfrac{\Phi}{\Phi^*}\right)^2}\exp\left(\dfrac{\varepsilon^{1+a}}{1 - \dfrac{\Phi}{\Phi^*}}\right) & \text{if}\quad \dfrac{\Phi}{\Phi^*} < 1 - \delta \\[2em]
0 & \text{if}\quad \dfrac{\Phi}{\Phi^*} \ge 1 - \delta
\end{cases} \tag{31}$$

The proof of Theorem 1 will consist of two limit passages, first $\delta \to 0$ then $\vartheta \to 0$.

Remark : One could try to extend the procedure of Vasseur and Yu in [36] and derive a control on $\Phi|u|^2 \log(1+|u|^2)$ but it seems not possible to get a uniform estimate with respect to ε or even δ. This is the reason why we need to keep the the turbulent term in the momentum equation (28b).

Organization of the paper

The rest of this article is coarsely divided in three parts. The first part concerns the proof of Theorem 1, namely the existence of weak solutions for what we call the "suspension model" (10a)–(10b) with singular viscosities μ_ε, λ_ε and singular pressure π_ε. The main part, corresponding to Theorem 2, consists of passing from solutions of this suspension model towards solutions of the two-phase system of granular type (24a)–(24e). Finally using this result we approximate the incompressible model (27a)–(27c) by an appropriate suspension system and prove therefore the existence of global weak solutions for (27a)–(27c) as stated in Theorem 3.

3   Existence of solutions to the suspension model

3.1   Global existence of κ-entropy solutions when ε, δ are fixed

We first need to prove the approximate system containing all the parameters ε, δ, admits global weak solutions. We recall in the following definition the notion of κ-entropy solutions for system (28a)–(28b).

Definition 2 (κ-entropy solutions for (28)) *Let $T > 0$, $\kappa \in (0,1)$, (Φ_δ, u_δ) is called a κ-entropy solution to system (28a)–(28b) if it satisfies*

- *the mass equation in the weak sense*

$$-\int_0^T \int_\Omega \Phi_\delta \partial_t \xi - \int_0^T \int_\Omega \Phi_\delta u_\delta \cdot \nabla \xi = \int_\Omega \Phi_\varepsilon^0 \xi(0) \qquad \forall \xi \in \mathcal{D}([0,T) \times \Omega) \qquad (32)$$

- *the momentum equation in the weak sense, $\forall \zeta \in (\mathcal{D}([0,T) \times \Omega))^2$*

$$-\int_0^T \int_\Omega \Phi_\delta u_\delta \cdot \partial_t \zeta - \int_0^T \int_\Omega (\Phi_\delta u_\delta \otimes u_\delta) : \nabla \zeta - \vartheta \int_0^T \int_\Omega p(\Phi_\delta) \text{div}\, \zeta$$
$$-\int_0^T \int_\Omega \pi_{\varepsilon,\delta}(\Phi_\delta) \text{div}\, \zeta + r \int_0^T \int_\Omega \Phi_\delta |u_\delta| u_\delta \cdot \zeta$$
$$+2 \int_0^T \int_\Omega \mu_{\varepsilon,\delta}(\Phi_\delta) \, \text{D}\,(u_\delta) : \nabla \zeta + \int_0^T \int_\Omega \lambda_{\varepsilon,\delta}(\Phi_\delta) \text{div}\,(u_\delta) \text{div}\, \zeta$$
$$= \int_\Omega m_\varepsilon^0 \cdot \zeta(0) \qquad (33)$$

- *the κ-entropy inequality*

$$
\sup_{t\in[0,T]} \int_\Omega \Phi_\delta \left(\frac{|u_\delta + 2\kappa\nabla\varphi_{\varepsilon,\delta}(\Phi_\delta)|^2}{2} + \kappa(1-\kappa)\frac{|2\nabla\varphi_{\varepsilon,\delta}(\Phi_\delta)|^2}{2} \right)
$$

$$
+ \sup_{t\in[0,T]} \int_\Omega \left(\vartheta\frac{\Phi_\delta^2}{2} + \Phi_\delta e_{\varepsilon,\delta}(\Phi_\delta) + r\mu_{\varepsilon,\delta}(\Phi_\delta) \right)
$$

$$
+ \kappa \int_0^T \int_\Omega \mu_{\varepsilon,\delta}(\Phi_r)| \mathrm{A}\,(u_\delta)|^2 + 2\kappa \int_0^T \int_\Omega \mu'_{\varepsilon,\delta}(\Phi_\delta) \left(\vartheta + \frac{\pi'_{\varepsilon,\delta}(\Phi_\delta)}{\Phi_\delta} \right) |\nabla\Phi_\delta|^2
$$

$$
+ (1-\kappa) \int_0^T \left[\int_\Omega \mu_{\varepsilon,\delta}(\Phi_\delta)| \mathrm{D}\,(u_\delta)|^2 + \int_\Omega \left(\mu'_{\varepsilon,\delta}(\Phi_\delta)\Phi_\delta - \mu_{\varepsilon,\delta}(\Phi_\delta) \right) |\mathrm{div}\, u_\delta|^2 \right]
$$

$$
+ r \int_0^T \int_\Omega \Phi_\varepsilon |u_\delta|^3 \leq C(r) \tag{34}
$$

where $\varphi_{\varepsilon,\delta}$ is such that

$$
\varphi'_{\varepsilon,\delta}(\Phi) = \frac{\mu'_{\varepsilon,\delta}(\Phi)}{\Phi}.
$$

Remark: $C(r)$ is a constant which depends only on r and on the initial data $(\Phi_\varepsilon^0, m_\varepsilon^0)$. As we will show later on, this constant will derive from a non-linear Gronwall inequality.

In [10], D. BRESCH, B. DESJARDINS and E. ZATORSKA base their construction of approximate solutions on an augmented approximate scheme satisfied by $(\Phi, w = u + 2\kappa\nabla\varphi(\Phi), v = 2\nabla\varphi(\Phi))$. In our framework this augmented system writes as

$$
\begin{cases}
\partial_t \Phi_\delta + \mathrm{div}\,(\Phi_\delta w_\delta) - 2\kappa\Delta\mu_{\varepsilon,\delta}(\Phi_\delta) = 0 & (35a) \\[2mm]
\partial_t(\Phi_\delta w_\delta) + \mathrm{div}\,(\Phi_\delta u_\delta \otimes w_\delta) - 2(1-\kappa)\mathrm{div}\,(\mu_{\varepsilon,\delta}(\Phi_\delta)\nabla w_\delta) - 2\kappa\mathrm{div}\,(\mu_{\varepsilon,\delta}(\Phi_\delta) \mathrm{A}\,(w_\delta)) \\
\quad + \vartheta\nabla p(\Phi_\delta) + \nabla\pi_{\varepsilon,\delta}(\Phi_\delta) + 4(1-\kappa)\kappa\mathrm{div}\,(\mu_{\varepsilon,\delta}(\Phi_\delta)\nabla^2\varphi_{\varepsilon,\delta}(\Phi_\delta)) \\
\quad - \nabla\big((\lambda_{\varepsilon,\delta}(\Phi_\delta) - 2\kappa(\mu'_{\varepsilon,\delta}(\Phi_\delta)\Phi_\delta - \mu_{\varepsilon,\delta}(\Phi_\delta)))\mathrm{div}\, u_\delta\big) + r\Phi_\delta|u_\delta|u_\delta = 0 & (35b) \\[2mm]
\partial_t(\Phi_\delta\nabla\varphi_{\varepsilon,\delta}(\Phi_\delta)) + \mathrm{div}\,(\Phi_\delta u_\delta \otimes \nabla\varphi_{\varepsilon,\delta}(\Phi_\delta)) - 2\kappa\mathrm{div}\,(\mu_{\varepsilon,\delta}(\Phi_\delta)\nabla^2\varphi_{\varepsilon,\delta}(\Phi_\delta)) \\
\quad + \mathrm{div}\,(\mu_{\varepsilon,\delta}(\Phi_\delta)\nabla^t w_\delta) + \nabla\big((\mu'_{\varepsilon,\delta}(\Phi_\delta)\Phi_\delta - \mu_{\varepsilon,\delta}(\Phi_\delta))\mathrm{div}\, u_\delta\big) = 0 & (35c) \\[2mm]
w_\delta = u_\delta + 2\kappa\,\nabla\varphi_{\varepsilon,\delta}(\Phi_\delta) & (35d)
\end{cases}
$$

Justification of the κ-entropy inequality : Following the steps of D. BRESCH, B. DESJARDINS and E. ZATORSKA [10], we multiply (35b) by $w_\delta = u_\delta + 2\kappa\nabla\varphi_{\varepsilon,\delta}(\Phi_\delta)$ and we combine it with

the equation (35c) tested by $4\kappa(1-\kappa)\nabla\varphi_{\varepsilon,\delta}(\Phi_\delta)$, we get

$$
\int_\Omega \left[\Phi_\delta \left(\frac{|u_\delta + 2\kappa\nabla\varphi_{\varepsilon,\delta}(\Phi_\delta)|^2}{2} + \kappa(1-\kappa)\frac{|2\nabla\varphi_{\varepsilon,\delta}(\Phi_\delta)|^2}{2} \right) \right](t)
$$

$$
+ \int_\Omega \left[\vartheta\frac{\Phi_\delta^2}{2} + \Phi_\delta e_{\varepsilon,\delta}(\Phi_\delta) \right](t) + r\int_0^t \int_\Omega \Phi_\delta |u_\delta|^3
$$

$$
+ 2\kappa \int_0^t \int_\Omega \mu_{\varepsilon,\delta}(\Phi_\delta)|A(u_\delta)|^2 + 2\kappa \int_0^t \int_\Omega \frac{\mu'_{\varepsilon,\delta}(\Phi_\delta)\pi'_{\varepsilon,\delta}(\Phi_\delta)}{\Phi_\delta}|\nabla\Phi_\delta|^2
$$

$$
+ 2(\kappa-1)\int_0^t \int_\Omega \mu_{\varepsilon,\delta}(\Phi_\delta)|\,\mathrm{D}\,(u_\delta)|^2 + 2(\kappa-1)\int_0^t \int_\Omega (\mu'_{\varepsilon,\delta}(\Phi_\delta)\Phi_\delta - \mu_{\varepsilon,\delta}(\Phi_\delta))(\mathrm{div}\,u_\delta)^2
$$

$$
= \int_\Omega \left[\Phi_\varepsilon^0 \left(\frac{\left|m_\varepsilon^0/\Phi_\varepsilon^0 + 2\kappa\nabla\varphi_{\varepsilon,\delta}(\Phi_\varepsilon^0)\right|^2}{2} + \kappa(1-\kappa)\frac{|2\nabla\varphi_{\varepsilon,\delta}(\Phi_\varepsilon^0)|^2}{2} \right) \right] \tag{36}
$$

$$
+ \int_\Omega \left[\vartheta\frac{(\Phi_\varepsilon^0)^2}{2} + \Phi_\varepsilon^0 e_{\varepsilon,\delta}(\Phi_\varepsilon^0) \right] - 2\kappa r\int_0^t \int_\Omega \Phi_\delta |u_\delta| u_\delta \cdot \nabla\varphi_{\varepsilon,\delta}(\Phi_\delta)
$$

In addition, the renormalized continuity equation writes

$$
\partial_t\big(\mu_{\varepsilon,\delta}(\Phi_\delta)\big) + \mathrm{div}\,(\mu_{\varepsilon,\delta}(\Phi_\delta)u_\delta) = -\lambda_{\varepsilon,\delta}(\Phi_\delta)\mathrm{div}\,u_\delta
$$

which gives, after multiplication by r and integration in time and space,

$$
r\int_\Omega \mu_{\varepsilon,\delta}(\Phi_\delta)(t) + r\int_0^t \int_\Omega \frac{\lambda_{\varepsilon,\delta}(\Phi_\delta)}{2}\mathrm{div}\,u_\delta = 0. \tag{37}
$$

Adding then (37) and (36)

$$
\int_\Omega \left[\Phi_\delta \left(\frac{|u_\delta + 2\kappa\nabla\varphi_{\varepsilon,\delta}(\Phi_\delta)|^2}{2} + \kappa(1-\kappa)\frac{|2\nabla\varphi_{\varepsilon,\delta}(\Phi_\delta)|^2}{2} \right) \right](t)
$$

$$
+ \int_\Omega \left[\vartheta\frac{\Phi_\delta}{2} + \Phi_\delta e_{\varepsilon,\delta}(\Phi_\delta) + r\mu_{\varepsilon,\delta}(\Phi_\delta) \right](t) + r\int_0^t \int_\Omega \Phi_\delta |u_\delta|^3
$$

$$
+ 2\kappa \int_0^t \int_\Omega \mu_{\varepsilon,\delta}(\Phi_\delta)|A(u_\delta)|^2 + 2\kappa \int_0^t \int_\Omega \frac{\mu'_{\varepsilon,\delta}(\Phi_\delta)\pi'_{\varepsilon,\delta}(\Phi_\delta)}{\Phi_\delta}|\nabla\Phi_\delta|^2
$$

$$
+ 2(\kappa-1)\int_0^t \int_\Omega \mu_{\varepsilon,\delta}(\Phi_\delta)|\,\mathrm{D}\,(u_\delta)|^2 + 2(\kappa-1)\int_0^t \int_\Omega (\mu'_{\varepsilon,\delta}(\Phi_\delta)\Phi_\delta - \mu_{\varepsilon,\delta}(\Phi_\delta))(\mathrm{div}\,u_\delta)^2
$$

$$
= \int_\Omega \left[\Phi_\varepsilon^0 \left(\frac{\left|m_\varepsilon^0/\Phi_\varepsilon^0 + 2\kappa\nabla\varphi_{\varepsilon,\delta}(\Phi_\varepsilon^0)\right|^2}{2} + \kappa(1-\kappa)\frac{|2\nabla\varphi_{\varepsilon,\delta}(\Phi_\varepsilon^0)|^2}{2} \right) \right]
$$

$$
+ \int_\Omega \left[\vartheta\frac{(\Phi_\varepsilon^0)^2}{2} + \Phi_\varepsilon^0 e_{\varepsilon,\delta}(\Phi_\varepsilon^0) \right] - r\int_0^t \int_\Omega \frac{\lambda_{\varepsilon,\delta}(\Phi_\delta)}{2}\mathrm{div}\,u_\delta
$$

$$
- 2\kappa r\int_0^t \int_\Omega \Phi_\delta |u_\delta| u_\delta \cdot \nabla\varphi_{\varepsilon,\delta}(\Phi_\delta) \tag{38}
$$

We now have to control the last two integrals of the right–hand side in the previous relation:

$$
J_1 = \frac{r}{2}\int_0^t \int_\Omega \lambda_{\varepsilon,\delta}(\Phi_\delta)\mathrm{div}\,u_\delta, \qquad J_2 = 2\kappa r\int_0^t \int_\Omega \Phi_\delta |u_\delta| u_\delta \cdot \nabla\varphi_{\varepsilon,\delta}(\Phi_\delta).
$$

Control of $J_1 = \frac{r}{2}\int_0^t \int_\Omega \lambda_{\varepsilon,\delta}(\Phi_\delta)\mathrm{div}\, u_\delta$. Unfortunatly it not possible to control uniformly with respect to all the parameters directly by the left–hand side of (38). The idea is to apply in a certain sense the operator $(-\Delta)^{-1}\mathrm{div}$ to the momentum equation where Δ^{-1} denotes the inverse operator of the Laplace operator. For each function f such that $\int_\Omega f = 0$ we denote $g = (-\Delta)^{-1}f$ the unique periodic function such that $-\Delta g = f$ and $\int_\Omega g = 0$. Therefore we can obtain the equality

$$
\begin{aligned}
r\int_0^t \int_\Omega \frac{\lambda_{\varepsilon,\delta}(\Phi_\delta)}{2}\mathrm{div}\, u_\delta &= \frac{r}{2}\int_\Omega (-\Delta)^{-1}\mathrm{div}\,(\Phi_\delta u_\delta)(t) - \frac{r}{2}\int_\Omega (-\Delta)^{-1}\mathrm{div}\,(m_\varepsilon^0) \\
&\quad - \frac{r}{2}\int_0^t \int_\Omega \Delta^{-1}\partial_i\partial_j(\Phi_\delta u_\delta^i u_\delta^j) \\
&\quad + \frac{r}{2}\vartheta \int_0^t \int_\Omega p(\Phi_\delta) + \frac{r}{2}\int_0^t \int_\Omega \pi_{\varepsilon,\delta}(\Phi_\delta) \\
&\quad + r\int_0^t \int_\Omega \Delta^{-1}\partial_i\partial_j\left[\mu_{\varepsilon,\delta}(\Phi_\delta)\left(\frac{\partial_i u_\delta^j + \partial_j u_\delta^i}{2}\right)\right] \\
&\quad + \frac{r^2}{2}\int_0^t \int_\Omega (-\Delta)^{-1}\mathrm{div}\,(\Phi_\delta |u_\delta| u_\delta) \\
&= \sum_{k=1}^7 I_k
\end{aligned}
\tag{39}
$$

In order to fully justify the previous equation, it suffices to take in the weak formulation (33) the test function

$$
\zeta_1(t)\nabla\Delta^{-1}(\zeta_2(x)), \quad \zeta_1 \in \mathcal{C}^\infty((0,T)), \quad \zeta_2 \in \mathcal{C}^\infty(\Omega), \quad \int_\Omega \zeta_2(x) = 0.
$$

We refer to [16] Section 2.2.6 for a similar computation and to Section 10.16 for properties of the singular operators involved in the previous equation.

Before studying each integral I_k of the previous equation let us explain how the norms of $\mu_{\varepsilon,\delta}(\Phi_\delta)$ are treated. By the Gagliardo–Nirenberg inequality, see for instance [29] (Theorem p.12 with $j = 0$ and $m = 1$ and Remark 5.), we have

$$
\|\mu_{\varepsilon,\delta}(\Phi_\delta)\|_{L^q(\Omega)} \le C\left(\|\mu_{\varepsilon,\delta}(\Phi_\delta)\|_{L^1(\Omega)}^\theta \|\nabla\mu_{\varepsilon,\delta}(\Phi_\delta)\|_{L^{\tilde{q}}(\Omega)}^{1-\theta} + \|\mu_{\varepsilon,\delta}(\Phi_\delta)\|_{L^1(\Omega)}\right)
$$

for all $\tilde{q} \in [1,2)$,

$$
q \in \left(1, \frac{2\tilde{q}}{2 - \tilde{q}}\right)
$$

and where

$$
\frac{1}{q} = \theta + \frac{1-\theta}{2\tilde{q}/(2-\tilde{q})}.
$$

We obtain then

$$
\begin{aligned}
\|\mu_{\varepsilon,\delta}(\Phi_\delta)\|_{L^q(\Omega)} &\le C_1\|\mu_{\varepsilon,\delta}(\Phi_\delta)\|_{L^1(\Omega)}^{\alpha_1} + C_2\left\|\frac{\nabla\mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}}\right\|_{L^2(\Omega)}\|\sqrt{\Phi_\delta}\|_{L^{2\tilde{q}/(2-\tilde{q})}(\Omega)} \\
&\le C_1\|\mu_{\varepsilon,\delta}(\Phi_\delta)\|_{L^1(\Omega)}^{\alpha_1} + \frac{C_2}{\nu}\left\|\frac{\nabla\mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}}\right\|_{L^2(\Omega)}^2 + C_2\nu\|\mu_{\varepsilon,\delta}(\Phi_\delta)\|_{L^{\tilde{q}/(2-\tilde{q})}(\Omega)}
\end{aligned}
$$

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If we consider now $\tilde{q}$ such that

$$q \in \left(\frac{\tilde{q}}{2 - \tilde{q}}, \frac{2\tilde{q}}{2 - \tilde{q}} \right)$$

and $\nu > 0$ small enough the last term can be absorbed by the left–hand side and replacing $\mu_{\varepsilon,\delta}(\Phi_\delta)$ by $r\mu_{\varepsilon,\delta}(\Phi_\delta)$ we deduce that

$$\|r\mu_{\varepsilon,\delta}(\Phi_\delta)\|_{L^q(\Omega)} \le C \left(\|r\mu_{\varepsilon,\delta}(\Phi_\delta)\|^{\alpha_1}_{L^1(\Omega)} + \|\frac{\nabla\mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}}\|^{\alpha_2}_{L^2(\Omega)} \right) \tag{40}$$

with α_1 and α_2 two positive constants depending only on q. We come back now to (39) and detail the control of each integral . Thanks to the regularity properties and conservation of mass we have for I_1 (and the same for I_2)

$$|I_1| \le r \int_\Omega \left| (-\Delta)^{-1} \mathrm{div}\, (\Phi_\delta u_\delta) \right| \le Cr\|\sqrt{\Phi_\delta}\|_{L^\infty L^2}\|\sqrt{\Phi_\delta}u_\delta\|_{L^\infty L^2} \le Cr\|\sqrt{\Phi_\delta}u_\delta\|_{L^\infty L^2}$$

which can be absorbed by the left–hand side provided that r is small enough.
Concerning the convective term in I_3, we use the estimate due to the drag and the control of $\Phi_\delta^{\gamma+1}$, this control coming from the splitting

$$\|\Phi_\delta^{\gamma+1}\|_{L^1(\Omega)} = \|\Phi_\delta^{\gamma+1}\left(\mathbf{1}_{\{\Phi_\delta \le \Phi^*/2\}} + \mathbf{1}_{\{\Phi_\delta \ge \Phi^*/2\}}\right)\|_{L^1(\Omega)}$$
$$\le C + C\|\Phi_\delta e_{\varepsilon,\delta}(\Phi_\delta)\|_{L^1(\Omega)}$$

Hence,

$$|I_3| \le r \int_0^t \int_\Omega \left| \Delta^{-1} \partial_i \partial_j (\Phi_\delta u_\delta^i u_\delta^j) \right| \le Cr\|\Delta^{-1}\partial_i\partial_j(\Phi_\delta|u_\delta|^2)\|_{L^1 L^{3(\gamma+1)/(3+2\gamma)}}$$
$$\le Cr\|\Phi_\delta^{1/3}\Phi_\delta^{2/3}|u_\delta|^2\|_{L^1 L^{3(\gamma+1)/(3+2\gamma)}}$$
$$\le Cr^{1/3}\|\Phi_\delta^{1/3}\|_{L^3 L^{3(\gamma+1)}}\|r^{2/3}\Phi_\delta^{2/3}|u_\delta|^2\|_{L^{3/2}L^{3/2}}$$
$$\le Cr^{1/3}\|\Phi_\delta\|^{1/3}_{L^1 L^{\gamma+1}}\|r^{1/3}\Phi_\delta^{1/3}u_\delta\|^2_{L^3 L^3}$$
$$\le Cr^{1/3}\|\Phi_\delta e_{\varepsilon,\delta}(\Phi_\delta)\|^{1/3}_{L^1 L^1}\|r^{1/3}\Phi_\delta^{1/3}u_\delta\|^2_{L^3 L^3}$$

Concerning the pressure terms, they have a positve sign and we do not need to control them. The integral of the drag can be controlled as follows

$$|I_7| \le r^2 \int_0^t \int_\Omega (-\Delta)^{-1}\mathrm{div}\,(\Phi_\delta|u_\delta|u_\delta)$$
$$\le Cr\|r^{1/3}\Phi_\delta^{1/3}r^{2/3}\Phi_\delta^{2/3}|u_\delta|^2\|_{L^1 L^q} \quad \text{for all} \quad q > 1$$
$$\le Cr\|r^{1/3}\mu_{\varepsilon,\delta}(\Phi_\delta)\|_{L^3 L^{3q/(3-2q)}}\|r^{2/3}\Phi_\delta^{2/3}|u_\delta|^2\|_{L^{3/2}L^{3/2}}$$
$$\le Cr\left(\|r\mu_{\varepsilon,\delta}(\Phi_\delta)\|^{\alpha_1}_{L^3 L^1} + \|\frac{\nabla\mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}}\|^{\alpha_2}_{L^3 L^2} \right) \|r^{1/3}\Phi_\delta^{1/3}u_\delta\|^2_{L^3 L^3}.$$

Finally, the last integral I_6, coming from the viscosity $\mu_{\varepsilon,\delta}(\Phi_\delta)\,\mathrm{D}\,(u_\delta)$, is the most difficult

$$I_6 = r \int_0^t \int_\Omega \Delta^{-1}\partial_i\partial_j \left[\mu_{\varepsilon,\delta}(\Phi_\delta)\left(\frac{\partial_i u_\delta^j + \partial_j u_\delta^i}{2} \right) \right].$$

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We see then that it is sufficient to control the gradient part, namely

$$r \int_0^t \int_\Omega \Delta^{-1} \partial_i \partial_j \left(\mu_{\varepsilon,\delta}(\Phi_\delta) \partial_i u_\delta^j \right),$$

the remaining term can be then treated exactly in the same way. Writing that

$$\mu_{\varepsilon,\delta}(\Phi_\delta) \partial_i u_\delta^j = \partial_i (\mu_{\varepsilon,\delta}(\Phi_\delta) u_\delta^j) - \partial_i (\mu_{\varepsilon,\delta}(\Phi_\delta)) u_\delta^j$$

we observe that the first term combined with the operator $\Delta^{-1} \partial_i \partial_j$ and the integration over Ω will give us 0. We get

$$
\begin{aligned}
r \int_0^t \int_\Omega & \Delta^{-1} \partial_i \partial_j \left(\mu_{\varepsilon,\delta}(\Phi_\delta) \partial_i u_\delta^j \right) \\
& \leq Cr \|\nabla \mu_{\varepsilon,\delta}(\Phi_\delta) u_\delta\|_{L^1 L^q} \\
& \leq Cr^{1-1/3-1/6} \|\frac{\nabla \mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}}\|_{L^3 L^2} \|r^{1/3} \Phi_\delta^{1/3} u_\delta\|_{L^3 L^3} \|r^{1/6} \Phi_\delta^{1/6}\|_{L^3 L^{6q/(6-5q)}} \\
& \leq Cr^{1/2} \|\frac{\nabla \mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}}\|_{L^3 L^2} \|r^{1/3} \Phi_\delta^{1/3} u_\delta\|_{L^3 L^3} \|r \mu_{\varepsilon,\delta}(\Phi_\delta)\|_{L^3 L^{q/(6-5q)}}^{1/6}
\end{aligned}
$$

for all $q \in (1, 6/5)$ and (40) yields

$$
\begin{aligned}
r \int_0^t \int_\Omega & \Delta^{-1} \partial_i \partial_j \left(\mu_{\varepsilon,\delta}(\Phi_\delta) \partial_i u_\delta^j \right) \\
& \leq Cr^{1/2} \left(\|r \mu_{\varepsilon,\delta}(\Phi_\delta)\|_{L^3 L^1}^{\alpha_1} + \|\frac{\nabla \mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}}\|_{L^3 L^2}^{\alpha_2} \right) \|r^{1/3} \Phi_\delta^{1/3} u_\delta\|_{L^3 L^3}.
\end{aligned}
$$

This concludes the control of the integral $\int_0^t \int_\Omega \lambda_{\varepsilon,\delta}(\Phi_\delta) \operatorname{div} u_\delta$ in (38).

Control of $J_2 = 2\kappa r \int_0^t \int_\Omega \Phi_\delta |u_\delta| u_\delta \cdot \nabla \varphi_{\varepsilon,\delta}(\Phi_\delta)$. It remains then in (38) to control the integral coming from the drag

$$
\begin{aligned}
I &= 2\kappa r \int_0^T \int_\Omega |u_\delta| u_\delta \cdot \nabla \mu_{\varepsilon,\delta}(\Phi_\delta) \\
&= -2\kappa r \int_0^T \int_\Omega \mu_{\varepsilon,\delta}(\Phi_\delta) |u_\delta| \operatorname{div} (u_\delta) - 2\kappa r \int_0^T \int_\Omega \mu_{\varepsilon,\delta}(\Phi_\delta) \frac{u_\delta^i}{|u_\delta|} u_\delta^j \partial_j u_\delta^i
\end{aligned}
$$

Splitting ∇u_δ between its symmetric and its skew-symmetric part, we get

$$
\begin{aligned}
|I| &\leq c(\Omega) \kappa r \int_0^T \int_\Omega |\mu_{\varepsilon,\delta}(\Phi_\delta)| |\nabla u_\delta| |u_\delta| \\
&\leq c(\Omega) \kappa r \int_0^T \int_\Omega |\mu_{\varepsilon,\delta}(\Phi_\delta)| |\operatorname{D}(u_\delta)| |u_\delta| + c(\Omega) \kappa r \int_0^T \int_\Omega |\mu_{\varepsilon,\delta}(\Phi_\delta)| |\operatorname{A}(u_\delta)| |u_\delta| \\
&\leq \frac{c(\Omega) \kappa r^{2/3}}{\sqrt{2(1-\kappa)}} \|\frac{\sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)}}{\Phi_\delta^{1/3}}\|_{L^6 L^6} \|\sqrt{2(1-\kappa)} \sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)} \operatorname{D}(u_\delta)\|_{L^2 L^2} \|r^{1/3} \Phi_\delta^{1/3} |u_\delta|\|_{L^3 L^3} \\
&\quad + c(\Omega) \sqrt{\kappa} r^{2/3} \|\frac{\sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)}}{\Phi_\delta^{1/3}}\|_{L^6 L^6} \|\sqrt{2\kappa} \sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)} \operatorname{A}(u_\delta)\|_{L^2 L^2} \|r^{1/3} \Phi_\delta^{1/3} u_\delta\|_{L^3 L^3} \\
&\leq C_1 r^{2/3} \|\sqrt{2(1-\kappa)} \sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)} \operatorname{D}(u_\delta)\|_{L^2 L^2}^2 + C_2 r^{2/3} \|\sqrt{2\kappa} \sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)} \operatorname{A}(u_\delta)\|_{L^2 L^2}^2 \\
&\quad + C_3 r^{2/3} \|r^{1/3} \Phi_\delta^{1/3} u_\delta\|_{L^3 L^3}^3 + C r^{2/3} \|\frac{\sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)}}{\Phi_\delta^{1/3}}\|_{L^6 L^6}^6 \qquad (41)
\end{aligned}
$$

For r small enough we can ensure that

$$\max\left(C_1, C_2, C_3\right) r^{2/3} \leq \frac{1}{2}$$

and absorb the first three terms of I by the left-hand side of (38). Splitting the last term into two parts

$$\|\frac{\sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)}}{\Phi_\delta^{1/3}}\|_{L^6 L^6} \leq \|\frac{\sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)}}{\Phi_\delta^{1/3}}\mathbf{1}_{\{\Phi_\delta \leq \Phi^*/2\}}\|_{L^6 L^6} + \|\frac{\sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)}}{\Phi_\delta^{1/3}}\mathbf{1}_{\{\Phi_\delta > \Phi^*/2\}}\|_{L^6 L^6}$$

we have on one hand

$$\frac{\sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)}}{\Phi_\delta^{1/3}}\mathbf{1}_{\{\Phi_\delta \leq \Phi^*/2\}} \leq \Phi_\delta^{1/6}\left[\sqrt{\frac{1}{\varepsilon}\left(\exp\left(\frac{\varepsilon^{1+a}}{2}\right)-1\right)}+1\right]\mathbf{1}_{\{\Phi_\delta \leq \Phi^*/2\}}$$

$$\leq C\Phi_\delta^{1/6}\mathbf{1}_{\{\Phi_\delta \leq \Phi^*/2\}}$$

which is bounded in $L^\infty((0,T)\times\Omega)$ and on the other hand,

$$\frac{\sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)}}{\Phi_\delta^{1/3}}\mathbf{1}_{\{\Phi_\delta > \Phi^*/2\}} \leq C\sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)}\mathbf{1}_{\{\Phi_\delta > \Phi^*/2\}}.$$

Using again (40)

$$\|\sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)}\mathbf{1}_{\{\Phi_\delta > \Phi^*/2\}}\|_{L^6 L^6} \leq C\left(\|\mu_{\varepsilon,\delta}(\Phi_\delta)\|_{L^{12} L^1}^{\alpha_1} + \|\frac{\nabla\mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}}\|_{L^{12} L^2}^{\alpha_2}\right).$$

Finally, coming back to (38), there exists $\alpha > 1$ and $\alpha_r > 0$ such that

$$\int_\Omega \left[\Phi_\delta\left(\frac{|u_\delta + 2\kappa\nabla\varphi_{\varepsilon,\delta}(\Phi_\delta)|^2}{2} + \kappa(1-\kappa)\frac{|2\varphi_{\varepsilon,\delta}(\Phi_\delta)|^2}{2}\right)\right](t)$$

$$+ \int_\Omega \left[\Phi_\delta e_{\varepsilon,\delta}(\Phi_\delta) + r\mu_{\varepsilon,\delta}(\Phi_\delta)\right](t) + \frac{r}{2}\int_0^t\int_\Omega \left(\vartheta p(\Phi_\delta) + \pi_{\varepsilon,\delta}(\Phi_\delta)\right) + \frac{r}{2}\int_0^t\int_\Omega \Phi_\delta |u_\delta|^3$$

$$+ \kappa\int_0^t\int_\Omega \mu_{\varepsilon,\delta}(\Phi_\delta)|A(u_\delta)|^2 + 2\kappa\int_0^t\int_\Omega \frac{\mu'_{\varepsilon,\delta}(\Phi_\delta)\pi'_{\varepsilon,\delta}(\Phi_\delta)}{\Phi_\delta}|\nabla\Phi_\delta|^2$$

$$+ (\kappa-1)\int_0^t\int_\Omega \mu_{\varepsilon,\delta}(\Phi_\delta)|\operatorname{D}(u_\delta)|^2 + (\kappa-1)\int_0^t\int_\Omega \left(\mu'_{\varepsilon,\delta}(\Phi_\delta)\Phi_\delta - \mu_{\varepsilon,\delta}(\Phi_\delta)\right)(\operatorname{div} u_\delta)^2$$

$$\leq \int_\Omega \left[\Phi_\varepsilon^0\left(\frac{|m_\varepsilon^0/\Phi_\varepsilon^0 + 2\kappa\nabla\varphi_{\varepsilon,\delta}(\Phi_\varepsilon^0)|^2}{2} + \kappa(1-\kappa)\frac{|2\nabla\varphi_{\varepsilon,\delta}(\Phi_\varepsilon^0)|^2}{2}\right)\right] + \int_\Omega \left[\vartheta\frac{(\Phi_\varepsilon^0)^2}{2} + \Phi_\varepsilon^0 e_{\varepsilon,\delta}(\Phi_\varepsilon^0)\right]$$

$$+ Cr^{\alpha_r}$$

$$\times \int_0^t \left(\int_\Omega \left[\Phi_\delta\left(\frac{|u_\delta + 2\kappa\nabla\varphi_{\varepsilon,\delta}(\Phi_\delta)|^2}{2} + \kappa(1-\kappa)\frac{|2\nabla\varphi_{\varepsilon,\delta}(\Phi_\delta)|^2}{2}\right) + \Phi_\delta e_{\varepsilon,\delta}(\Phi_\delta) + r\mu_{\varepsilon,\delta}(\Phi_\delta)\right]\right)^\alpha$$

For r small enough we can use a nonlinear generalization of the Gronwall Lemma on the interval $[0,T]$, see for instance [4] or [5] (Lemma II.4.12 p.90, with $f(y) = r^{\alpha_r} y^\alpha$).
Hence we have closed the *kappa*-entropy inequality which allows us to use the existence result of D. BRESCH, B. DESJARDINS and E. ZATORSKA [10].

Proposition 1 *Let $T > 0$, ε, δ be fixed, there exists $r > 0$, depending only on T, such that there exists a global κ-entropy solution to system (28). In particular we have the following regularities*

$$\sqrt{\Phi_\delta}|w_\delta| \in L^\infty(0,T;L^2(\Omega)), \quad \sqrt{\Phi_\delta}|\nabla\varphi_{\varepsilon,\delta}(\Phi_\delta)| \in L^\infty(0,T;L^2(\Omega)) \tag{42}$$

$$\Phi_\delta e_{\varepsilon,\delta}(\Phi_\delta) \in L^\infty(0,T;L^1(\Omega)), \qquad \Phi_\delta \in L^\infty(0,T;L^2(\Omega)) \tag{43}$$

$$r\mu_{\varepsilon,\delta}(\Phi_\delta) \in L^\infty(0,T;L^1(\Omega)), \quad r\Phi_\delta|u_\delta|^3 \in L^1((0,T)\times\Omega) \tag{44}$$

$$\sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)}\nabla u_\delta \in L^2((0,T)\times\Omega), \quad \sqrt{\lambda_{\varepsilon,\delta}(\Phi_\delta)}\mathrm{div}\,u_\delta \in L^2((0,T)\times\Omega) \tag{45}$$

$$\sqrt{\mu'_{\varepsilon,\delta}(\Phi_\delta)\left(\vartheta + 2\frac{\pi'_{\varepsilon,\delta}(\Phi_\delta)}{\Phi_\delta}\right)}\nabla\Phi_\delta \in L^2((0,T)\times\Omega). \tag{46}$$

3.2 Proof of Theorem 1: Existence of weak solutions for suspension model (10)

We aim here at proving Theorem 1 by letting the parameters δ and ϑ go to 0.

More precisely, we try in this section to derive the uniform controls dealing with viscosities and pressures which become singular as $\delta \to 0$ and using the friction term to ensure the compactness of the approximate solutions. Passing to the limit $\delta \to 0$ in the equations, we prove that the limit volume fraction satisfies the maximal volume fraction constraint

$$0 \le \Phi_{\varepsilon,\vartheta}(t,x) \le \Phi^* \quad \text{a.e. on } (0,T)\times\Omega$$

Finally we perform the limit passage $\vartheta \to 0$ which means that we eliminate the artificial pressure $\vartheta\nabla\frac{\Phi^2}{2}$. This step does not present additional difficulty and will be briefly explained in the final remark.

Estimates

We recall that the singular terms write as

$$\pi_{\varepsilon,\delta}(\Phi_\delta) = \begin{cases} \dfrac{\Phi_\delta}{\varepsilon}\left(\dfrac{\Phi_\delta}{\Phi^*}\right)^\gamma\left(\exp\left(\dfrac{\varepsilon^{1+a}}{1-\dfrac{\Phi_\delta}{\Phi^*}}\right) - 1\right) & \text{if } \dfrac{\Phi_\delta}{\Phi^*} \le 1-\delta \\[3em] \dfrac{\Phi_\delta}{\varepsilon}\left(\dfrac{\Phi_\delta}{\Phi^*}\right)^\gamma\left(\exp\left(\dfrac{\varepsilon^{1+a}}{\delta}\right) - 1\right) & \text{if } \dfrac{\Phi_\delta}{\Phi^*} > 1-\delta \end{cases}$$

$$\mu_{\varepsilon,\delta}(\Phi_\delta) = \begin{cases} \dfrac{\Phi_\delta}{\varepsilon}\left(\exp\left(\dfrac{\varepsilon^{1+a}}{1-\dfrac{\Phi_\delta}{\Phi^*}}\right) - 1\right) + \Phi_\delta & \text{if } \dfrac{\Phi_\delta}{\Phi^*} \le 1-\delta \\[3em] \dfrac{\Phi_\delta}{\varepsilon}\left(\exp\left(\dfrac{\varepsilon^{1+a}}{\delta}\right) - 1\right) + \Phi_\delta & \text{if } \dfrac{\Phi_\delta}{\Phi^*} > 1-\delta \end{cases}$$

$$\lambda_{\varepsilon,\delta}(\Phi_\delta) = 2((\mu_{\varepsilon,\delta})'_+(\Phi_\delta)\Phi_\delta - \mu_{\varepsilon,\delta}(\Phi_\delta))$$

$$= \begin{cases} 2\varepsilon^a \dfrac{\Phi_\delta^2}{\Phi^* \left(1 - \dfrac{\Phi_\delta}{\Phi^*}\right)^2} \exp\left(\dfrac{\varepsilon^{1+a}}{1 - \dfrac{\Phi_\delta}{\Phi^*}}\right) & \text{if} \quad \dfrac{\Phi_\delta}{\Phi^*} < 1 - \delta \\[2em] 0 & \text{if} \quad \dfrac{\Phi_\delta}{\Phi^*} \geq 1 - \delta \end{cases}$$

Control of Φ_δ. Thanks to the κ-entropy inequality and to the bound $\mu'_{\varepsilon,\delta}(\Phi_\delta) \geq 1$ we ensure that

$$\nabla\sqrt{\Phi_\delta} \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega))$$

and then that Φ_δ is bounded in $L^\infty(0,T;L^p(\Omega))$ for all $p \in [1,+\infty)$. Moreover

$$\|\sqrt{\vartheta}\sqrt{\mu'_{\varepsilon,\delta}(\Phi_\delta)}|\nabla\Phi_\delta|\|_{L^2L^2} \leq C$$

which means that $\nabla\Phi_\delta$ is bounded in $L^2((0,T) \times \Omega)$.

Control of $\Phi_\delta u_\delta$. For the momentum we directly have $\Phi_\delta u_\delta = \sqrt{\Phi_\delta}\sqrt{\Phi_\delta}u_\delta$ bounded in $L^\infty(0,T;L^q(\Omega))$ for all $q \in [1,2)$ and $\nabla(\Phi_\delta u_\delta)$ bounded in $L^2(0,T;L^1(\Omega))$ by writing that

$$\nabla(\Phi_\delta u_\delta) = \sqrt{\Phi_\delta}\sqrt{\Phi_\delta}\nabla u_\delta + 2\sqrt{\Phi_\delta}u_\delta \otimes \nabla\sqrt{\Phi_\delta}.$$

In addition, the drag contribution the κ-entropy inequality provides

$$\Phi_\delta u_\delta = \Phi_\delta^{2/3}\Phi_\delta^{1/3}u_\delta \quad \text{bounded in} \quad L^3(0,T;L^q(\Omega)) \quad \forall q \in [1,3).$$

Controls of the viscosities. Thanks to the κ-entropy we control uniformly $\sqrt{\Phi_\delta}\nabla\varphi_{\varepsilon,\delta}(\Phi_\delta)$ in $L^\infty(0,T;L^2(\Omega))$. If we set $V_{\varepsilon,\delta}(\Phi_\delta)$ such that

$$V'_{\varepsilon,\delta}(\Phi_\delta) = \sqrt{\Phi_\delta}\varphi'_{\varepsilon,\delta}(\Phi_\delta) = \frac{\mu'_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}} \tag{47}$$

we deduce that $\nabla V_{\varepsilon,\delta}(\Phi_\delta)$ is bounded in $L^\infty(0,T;L^2(\Omega))$, coming back to the viscosity $\mu_{\varepsilon,\delta}(\Phi_\delta)$, using the bound $L^\infty(0,T;L^p(\Omega))$, $p < +\infty$, on Φ_δ we obtain

$$\nabla\mu_{\varepsilon,\delta}(\Phi_\delta) = \sqrt{\Phi_\delta}\nabla V_{\varepsilon,\delta}(\Phi_\delta) \quad \text{bounded in} \quad L^\infty(0,T;L^q(\Omega)), \ \forall q \in [1,2). \tag{48}$$

Since we control $\mu_{\varepsilon,\delta}(\Phi_\delta)$ uniformly in $L^\infty(0,T;L^1(\Omega))$ we deduce that

$$\mu_{\varepsilon,\delta}(\Phi_\delta) \quad \text{bounded in} \quad L^\infty(0,T;L^p(\Omega)), \ \forall p \in [1,+\infty). \tag{49}$$

$$\mu_{\varepsilon,\delta}(\Phi_\delta)\,\mathrm{D}(u_\delta) \quad \text{bounded in} \quad L^2(0,T;L^q(\Omega)), \ \forall q \in [1,2). \tag{50}$$

We next bound the other viscosity coefficient $\lambda_{\varepsilon,\delta}(\Phi_\delta)$ by comparison with

$$\mu^1_{\varepsilon,\delta}(\Phi_\delta) = \begin{cases} \dfrac{\Phi_\delta}{\varepsilon}\left(\exp\left(\dfrac{\varepsilon^{1+a}}{1 - \dfrac{\Phi_\delta}{\Phi^*}}\right) - 1\right) & \text{if} \quad \dfrac{\Phi_\delta}{\Phi^*} < 1 - \delta \\[2em] \dfrac{\Phi_\delta}{\varepsilon}\left(\exp\left(\dfrac{\varepsilon^{1+a}}{1-\delta}\right) - 1\right) & \text{if} \quad \dfrac{\Phi_\delta}{\Phi^*} \geq 1 - \delta \end{cases}, L^\infty(0,T;L^p(\Omega)) \quad p < +\infty$$

Indeed, the ratio for $\Phi_\delta/\Phi^* \leq 1 - \delta$ reads as

$$\frac{\lambda_{\varepsilon,\delta}(\Phi_\delta)}{\left(\mu^1_{\varepsilon,\delta}(\Phi_\delta)\right)^2} = \frac{2\varepsilon^a \Phi_\delta^2 \exp\left(\dfrac{\varepsilon^{1+a}}{1 - \Phi_\delta/\Phi^*}\right)}{\Phi^*(1 - \Phi_\delta/\Phi^*)^2} \times \frac{\varepsilon^2}{\Phi_\delta^2 \left(\exp\left(\dfrac{\varepsilon^{1+a}}{1 - \Phi_\delta/\Phi^*}\right) - 1\right)^2}.$$

Using the fact that

$$\exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi_\delta/\Phi^*}\right) - 1 = \exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi_\delta/\Phi^*}\right)\left(1 - \exp\left(-\frac{\varepsilon^{1+a}}{1 - \Phi_\delta/\Phi^*}\right)\right)$$

$$\geq \exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi_\delta/\Phi^*}\right)\left(1 - \exp\left(-\varepsilon^{1+a}\right)\right)$$

we get then

$$\frac{\lambda_{\varepsilon,\delta}(\Phi_\delta)}{\left(\mu^1_{\varepsilon,\delta}(\Phi_\delta)\right)^2} \leq C(\varepsilon)\varepsilon^{-a} \frac{\varepsilon^{2(1+a)}}{(1 - \Phi_\delta/\Phi^*)^2} \exp\left(-\frac{\varepsilon^{1+a}}{1 - \Phi_\delta/\Phi^*}\right)$$

$$\leq C(\varepsilon)$$

because $X \mapsto X^2 \exp(-X)$ is a bounded function on $(0, +\infty)$. Since ε is fixed at this stage, we can conclude to the control of $\lambda_{\varepsilon,\delta}$

$$\lambda_{\varepsilon,\delta}(\Phi_\delta) \quad \text{bounded in} \quad L^\infty(0,T;L^p(\Omega)), \ \forall p \in [1, +\infty) \tag{51}$$

and in addition

$$\frac{\lambda_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)}} = \frac{\lambda_{\varepsilon,\delta}(\Phi_\delta)}{\mu_{\varepsilon,\delta}(\Phi_\delta)^2}\mu_{\varepsilon,\delta}(\Phi_\delta)^{3/2} \quad \text{bounded in} \quad L^\infty(0,T;L^p(\Omega)), \ \forall p \in [1, +\infty). \tag{52}$$

To pass to the limit in the diffusion terms written as in (22)–(23), we need also to control the quantity $\dfrac{\mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}}$. Let $\underline{\Phi} \in (0,1)$, on the set $\{\Phi_\delta/\Phi^* \geq \underline{\Phi}\}$,

$$\frac{\mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}} \leq \frac{\mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\underline{\Phi}}\sqrt{\Phi^*}}$$

is bounded in $L^\infty(0,T;L^p(\Omega))$, $p < \infty$. On the other set $\{\Phi_\delta/\Phi^* \leq \underline{\Phi}\}$, Φ_δ is far from Φ^* and then

$$\frac{\mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}} \leq \frac{\sqrt{\Phi_\delta}}{\varepsilon}\left(\exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi_\delta/\Phi^*}\right) - 1\right) + \sqrt{\Phi_\delta} \leq C\sqrt{\Phi_\delta}$$

which is still bounded in $L^\infty(0,T;L^p(\Omega))$ for all $p < \infty$. Therefore, in both cases

$$\frac{\mu_{\varepsilon,\delta}}{\sqrt{\Phi_\delta}} \quad \text{is bounded in} \quad L^\infty(0,T;L^p(\Omega)), \quad p \in [1, +\infty). \tag{53}$$

Control of the singular pressure. Since we have the relation

$$\pi_{\varepsilon,\delta}(\Phi_\delta) = \left(\frac{\Phi_\delta}{\Phi^*}\right)^\gamma \mu^1_{\varepsilon,\delta}(\Phi_\delta)$$

We can deduce directly controls on the singular pressure

$$\pi_{\varepsilon,\delta}(\Phi_\delta) \quad \text{is bounded in} \quad L^\infty(0,T;L^p(\Omega)), \quad p \in [1,+\infty). \tag{54}$$

and

$$\nabla \pi_{\varepsilon,\delta}(\Phi_\delta) = \frac{\gamma \Phi_\delta^{\gamma-1}}{(\Phi^*)^\gamma} \mu_{\varepsilon,\delta}^1(\Phi_\delta) \nabla \Phi_\delta + \left(\frac{\Phi_\delta}{\Phi^*}\right)^\gamma \nabla \mu_{\varepsilon,\delta}^1(\Phi_\delta)$$

$$\text{bounded in} \quad L^2(0,T;L^q(\Omega)), \, q \in [1,2). \tag{55}$$

Remark: Compared to the work done with constant viscosities in [31], the singular viscosities via the κ-entropy estimate (34) provide directly an uniform control of the singular pressure without additional estimates using the Bogovskii operator. In addition, we have much more integrability in the present case thanks to the κ-entropy which controls finally $\nabla \mu_{\varepsilon,\delta}(\Phi)$ and consequently all the powers of $\mu_{\varepsilon,\delta}(\Phi)$ and $\pi_{\varepsilon,\delta}(\Phi)$. In comparison, with constant viscosities we only get $\pi_{\varepsilon,\delta}(\Phi)$ bounded in $L^1((0,T) \times \Omega)$ which forces us to derive an additionnal estimate in order to prove the equi-integrability of the sequence.

Convergences

Following the classical steps of the stability of weak solutions of Navier-Stokes equations with degenerate viscosities (see for instance [28]), we prove first that the volume fraction Φ_δ converges to $\Phi_{\varepsilon,\vartheta}$ a.e. and in $\mathcal{C}([0,T];L^p(\Omega))$ for all $p \in [1,+\infty)$ thanks to the Aubin-Lions-Simon lemma (see [34]). This convergence leads then to the strong convergence of the pressures $p(\Phi_\delta)$ and $\pi_{\varepsilon,\delta}(\Phi_\delta)$ in $\mathcal{C}([0,T],L^1(\Omega))$.

Using again the Aubin-Lions-Simon lemma, we get the strong convergence in $L^2(0,T;L^q(\Omega))$ for all $q \in [1,2)$ and the convergence a.e. of $m_\delta = \Phi_\delta u_\delta$ towards some m. Since $m_\delta/\sqrt{\Phi_\delta}$ is bounded in $L^\infty(0,T;L^2(\Omega))$, Fatou's lemma provides

$$\int_\Omega \liminf_\delta \frac{m_\delta^2}{\Phi_\delta} < +\infty$$

which implies in particular that $m = 0$ a.e. on the vacuum set $\{\Phi_{\varepsilon,\vartheta} = 0\}$. Therefore we can define a limit velocity $u_{\varepsilon,\vartheta}$ such that

$$u_{\varepsilon,\vartheta}(t,x) = \begin{cases} \dfrac{m(t,x)}{\Phi_{\varepsilon,\vartheta}(t,x)} & \text{if} \quad \Phi_{\varepsilon,\vartheta} \neq 0 \\ 0 & \text{if} \quad \Phi_{\varepsilon,\vartheta} = 0 \end{cases}$$

Finally $\Phi_\delta u_\delta$ converges a.e. and strongly to $\Phi_{\varepsilon,\vartheta} u_{\varepsilon,\vartheta}$ in $L^2((0,T) \times \Omega)$.

At the limit we recover the maximal volume fraction constraint

Lemma 1 *At the limit $\delta \to 0$, we have* meas $\{(t,x) \, : \, \Phi_{\varepsilon,\vartheta}(t,x) \geq \Phi^*\} = 0$.

Proof. This result is based on the control of the singular potentiel energy when $\delta > 0$

$$\Phi_\delta e_{\varepsilon,\delta}(\Phi_\delta)\mathbf{1}_{\{\Phi_\delta/\Phi^* \geq 1-\delta\}}$$

$$\geq \Phi_\delta \left(\int_0^{\Phi^*(1-\delta)} \frac{\pi_{\varepsilon,\delta}(s)}{s^2} \,\mathrm{d}s \right) \mathbf{1}_{\{\Phi_\delta/\Phi^* \geq 1-\delta\}}$$

$$\geq \frac{\Phi_\delta}{(\Phi^*)^2(1-\delta)^2} \left(\int_0^{\Phi^*(1-\delta)} \pi_{\varepsilon,\delta}(s)\,\mathrm{d}s \right) \mathbf{1}_{\{\Phi_\delta/\Phi^* \geq 1-\delta\}}$$

$$\geq \frac{\mathbf{1}_{\{\Phi_\delta/\Phi^* \geq 1-\delta\}}}{\varepsilon\Phi^*(1-\delta)} \left(\int_0^{\Phi^*(1-\delta)} s \left(\frac{s}{\Phi^*}\right)^\gamma \left(\exp\left(\frac{\varepsilon^{1+a}}{1-s/\Phi^*}\right) - 1 \right) \mathrm{d}s \right)$$

$$\geq \frac{\mathbf{1}_{\{\Phi_\delta/\Phi^* \geq 1-\delta\}}}{\varepsilon\Phi^*(1-\delta)} \left(\int_0^{\Phi^*(1-\delta)} s \left(\frac{s}{\Phi^*}\right)^\gamma \left(\frac{\varepsilon^{1+a}}{1-s/\Phi^*} \right) \mathrm{d}s \right)$$

$$\geq \frac{\varepsilon^a \mathbf{1}_{\{\Phi_\delta/\Phi^* \geq 1-\delta\}}}{1-\delta} \int_0^{1-\delta} \Phi^* \frac{\tau^{\gamma+1}}{1-\tau} \,\mathrm{d}\tau$$

$$\geq \left(C_\varepsilon^1(-\log\delta) - C_\varepsilon^2 \right) \mathbf{1}_{\{\Phi_\delta/\Phi^* \geq 1-\delta\}}$$

Integrating over Ω and letting δ go to 0 we recover at the limit

$$\mathrm{meas}\left\{(t,x) \,:\, \Phi_{\varepsilon,\vartheta}(t,x) \geq \Phi^* \right\} = 0.$$

We can also prove the strong convergence of the singular terms $\mu_{\varepsilon,\delta}(\Phi_\delta)$, $\pi_{\varepsilon,\delta}(\Phi_\delta)$ and $\lambda_{\varepsilon,\delta}(\Phi_\delta)$ in $L^p((0,T) \times \Omega)$ for all $p \in [1,+\infty)$. Indeed we have previously seen that $\mu_{\varepsilon,\delta}(\Phi_\delta)$, $\pi_{\varepsilon,\delta}(\Phi_\delta)$ and $\lambda_{\varepsilon,\delta}(\Phi_\delta)$ are bounded in $L^p((0,T) \times \Omega)$ for all $p \in [1,+\infty)$. Besides Φ_δ converges a.e. and strongly to $\Phi_{\varepsilon,\vartheta}$ and we ensure that $\mathrm{meas}\left\{(t,x) \,:\, \Phi_{\varepsilon,\vartheta}(t,x) \geq \Phi^* \right\} = 0$. Therefore we guarantee that $\mu_{\varepsilon,\delta}(\Phi_\delta)$, $\pi_{\varepsilon,\delta}(\Phi_\delta)$ and $\lambda_{\varepsilon,\delta}(\Phi_\delta)$ converge a.e. towards $\mu_\varepsilon(\Phi_{\varepsilon,\vartheta})$, $\pi_\varepsilon(\Phi_{\varepsilon,\vartheta})$ and $\lambda_\varepsilon(\Phi_{\varepsilon,\vartheta})$ respectively. The Dominated Convergence Theorem finally provides the strong convergence of $\mu_{\varepsilon,\delta}$, $\pi_{\varepsilon,\delta}$ and $\lambda_{\varepsilon,\delta}$ in $L^p((0,T) \times \Omega)$ for all $p \in [1,+\infty)$.

Convergence in the drag term. We want to show that

$$\int_0^T \int_\Omega |\Phi_\delta|u_\delta|u_\delta - \Phi_{\varepsilon,\vartheta}|u_{\varepsilon,\vartheta}|u_{\varepsilon,\vartheta}| \longrightarrow 0$$

For that purpose, we introduce $R > 0$ and split the previous integral into three parts

$$\int_0^T \int_\Omega \left| \Phi_\delta|u_\delta|u_\delta - \Phi_{\varepsilon,\vartheta}|u_{\varepsilon,\vartheta}|u_{\varepsilon,\vartheta} \right| \mathrm{d}x\,\mathrm{d}t$$

$$\leq \int_0^T \int_\Omega \left| \Phi_\delta|u_\delta|u_\delta\mathbf{1}_{\{|u_\delta|\leq R\}} - \Phi_{\varepsilon,\vartheta}|u_{\varepsilon,\vartheta}|u_{\varepsilon,\vartheta}\mathbf{1}_{\{|u_{\varepsilon,\vartheta}|\leq R\}} \right| \mathrm{d}x\,\mathrm{d}t$$

$$+ \int_0^T \int_\Omega \left| \Phi_\delta|u_\delta|u_\delta\mathbf{1}_{\{|u_\delta|\geq R\}} \right| \mathrm{d}x\,\mathrm{d}t$$

$$+ \int_0^T \int_\Omega \left| \Phi_{\varepsilon,\vartheta}|u_{\varepsilon,\vartheta}|u_{\varepsilon,\vartheta}\mathbf{1}_{\{|u_{\varepsilon,\vartheta}|\geq R\}} \right| \mathrm{d}x\,\mathrm{d}t$$

First we have $\Phi_\delta|u_\delta|u_\delta = \Phi_\delta^{-1}\Phi_\delta^2|u_\delta|u_\delta$ which converges a.e. to $\Phi_{\varepsilon,\vartheta}^{-1}\Phi_{\varepsilon,\vartheta}^2|u_{\varepsilon,\vartheta}|u_{\varepsilon,\vartheta}$ on the set $\{\Phi_{\varepsilon,\vartheta} > 0\}$ thanks to the convergence a.e. on $(0,T) \times \Omega$ of $\Phi_\delta u_\delta$ and Φ_δ. In addition

$$\left| \Phi_\delta|u_\delta|u_\delta\mathbf{1}_{\{|u_\delta|\leq R\}} \right| \leq R^2\Phi_\delta \to 0 \quad \text{on} \quad \{\Phi_{\varepsilon,\vartheta} = 0\}$$

Therefore we get the convergence a.e. of $\Phi_\delta |u_\delta| u_\delta$ to $\Phi_{\varepsilon,\vartheta} |u_{\varepsilon,\vartheta}| u_{\varepsilon,\vartheta}$ and the Dominated Convergence Theorem gives the convergence to 0 of the first integral,

$$\int_0^T \int_\Omega \left| \Phi_\delta |u_\delta| u_\delta \mathbf{1}_{\{|u_\delta| \le R\}} - \Phi_{\varepsilon,\vartheta} |u_{\varepsilon,\vartheta}| u_{\varepsilon,\vartheta} \mathbf{1}_{\{|u_{\varepsilon,\vartheta}| \le R\}} \right| \, \mathrm{d}x \, \mathrm{d}t \to 0.$$

Concerning the two remaining integrals we use the control given by the κ-entropy and we write

$$\int_0^T \int_\Omega \left| \Phi_\delta |u_\delta| u_\delta \mathbf{1}_{\{|u_\delta| \ge R\}} \right| \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega \left| \Phi_{\varepsilon,\vartheta} |u_{\varepsilon,\vartheta}| u_{\varepsilon,\vartheta} \mathbf{1}_{\{|u_{\varepsilon,\vartheta}| \ge R\}} \right| \, \mathrm{d}x \, \mathrm{d}t$$

$$\le \frac{1}{R} \left(\int_0^T \int_\Omega \Phi_\delta |u_\delta|^3 \mathbf{1}_{\{|u_\delta| \ge R\}} \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega \Phi_{\varepsilon,\vartheta} |u_{\varepsilon,\vartheta}|^3 \mathbf{1}_{\{|u_{\varepsilon,\vartheta}| \ge R\}} \, \mathrm{d}x \, \mathrm{d}t \right)$$

$$\le \frac{C}{R}$$

Letting R go to $+\infty$, we obtain the strong convergence in $L^1((0,T) \times \Omega)$ of the turbulent drag term towards $\Phi_{\varepsilon,\vartheta} |u_{\varepsilon,\vartheta}| u_{\varepsilon,\vartheta}$.

Convergence in the convective term. For $\sqrt{\Phi_\delta} u_\delta$ we develop the same idea as for the turbulent drag term and we decompose the integral between the small and the large velocities

$$\int_0^T \int_\Omega \left| \sqrt{\Phi_\delta} u_\delta - \sqrt{\Phi_{\varepsilon,\vartheta}} u_{\varepsilon,\vartheta} \right|^2 \, \mathrm{d}x \, \mathrm{d}t$$

$$\le \int_0^T \int_\Omega \left| \sqrt{\Phi_\delta} u_\delta \mathbf{1}_{\{|u_\delta| \le R\}} - \sqrt{\Phi_{\varepsilon,\vartheta}} u_{\varepsilon,\vartheta} \mathbf{1}_{\{|u_{\varepsilon,\vartheta}| \le R\}} \right|^2 \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^T \int_\Omega \left| \sqrt{\Phi_\delta} u_\delta \mathbf{1}_{\{|u_\delta| \ge R\}} \right|^2 \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^T \int_\Omega \left| \sqrt{\Phi_{\varepsilon,\vartheta}} u_{\varepsilon,\vartheta} \mathbf{1}_{\{|u_{\varepsilon,\vartheta}| \ge R\}} \right|^2 \, \mathrm{d}x \, \mathrm{d}t$$

As previously we can show the convergence a.e. of $\sqrt{\Phi_\delta} u_\delta$ to $\sqrt{\Phi_{\varepsilon,\vartheta}} u_{\varepsilon,\vartheta}$ and the Dominated Convergence Theorem gives the convergence to 0 of the first integral. For the two last integrals we observe that we can split $\Phi |u|^2$ as

$$\Phi |u|^2 = \Phi^{1/3} \Phi^{2/3} |u|^2$$

and by the Hölder inequality with $1 < q < 3/2$ we get

$$\int_0^T \int_\Omega \Phi |u|^2 \mathbf{1}_{\{|u_\delta| \ge R\}} \, \mathrm{d}x \, \mathrm{d}t$$

$$\le \left(\int_0^T \int_\Omega \Phi^{q'/3} \, \mathrm{d}x \, \mathrm{d}t \right)^{1/q'} \left(\int_0^T \int_\Omega \Phi^{2q/3} |u|^{2q} \, \mathrm{d}x \, \mathrm{d}t \right)^{1/q}$$

$$\le \frac{C}{R^{(3-2q)/q}} \left(\int_0^T \int_\Omega \Phi |u|^3 \, \mathrm{d}x \, \mathrm{d}t \right)^{1/q} \xrightarrow[R \to \infty]{} 0$$

We conclude that

$$\sqrt{\Phi_\delta} u_\delta \quad \text{converges strongly to} \quad \sqrt{\Phi_{\varepsilon,\vartheta}} u_{\varepsilon,\vartheta} \quad \text{in} \quad L^2((0,T) \times \Omega).$$

Convergence in the diffusion terms. Since $\mu_{\varepsilon,\delta}(\Phi_\delta)\,\mathrm{D}\,(u_\delta)$ and $\lambda_{\varepsilon,\delta}(\Phi_\delta)\mathrm{div}\,u_\delta$ are bounded in $L^2(0,T;L^q(\Omega))$ for all $q \in [1,2)$, we deduce that they converge weakly in $L^2(0,T;L^q(\Omega))$ for all $q \in [1,2)$ towards $\overline{\mu_{\varepsilon,\delta}(\Phi)\,\mathrm{D}\,(u)}$ and $\overline{\lambda_{\varepsilon,\delta}(\Phi)\mathrm{div}\,u}$ respectively.

Remember that the diffusion terms make sense in the weak formulation of the momentum equation if they are written under the form

$$2\int_0^T\int_\Omega \mu_{\varepsilon,\delta}(\Phi_\delta)\,\mathrm{D}\,(u_\delta):\nabla\zeta = -\int_0^T\int_\Omega \sqrt{\Phi_\delta}u_\delta^j\left(\frac{\partial_i\mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}}\partial_i\zeta^j + \frac{\mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}}\partial_{ii}^2\zeta^j\right)$$
$$-\int_0^T\int_\Omega \sqrt{\Phi_\delta}u_\delta^i\left(\frac{\partial_j\mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}}\partial_i\zeta^j + \frac{\mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}}\partial_{ij}^2\zeta^j\right) \quad (56)$$

For the first integral we have the strong convergence in $L^2((0,T)\times\Omega)$ of $\sqrt{\Phi_\delta}u_\delta$ towards $\sqrt{\Phi_{\varepsilon,\vartheta}}u_{\varepsilon,\vartheta}$ and the weak convergence in $L^2((0,T)\times\Omega)$ of $\frac{\mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}}$ towards $\frac{\mu_\varepsilon(\Phi_{\varepsilon,\vartheta})}{\sqrt{\Phi_{\varepsilon,\vartheta}}}$ thanks to the control (53).

Recall that we define $V_{\varepsilon,\delta}$ such that (47)which allows us to write for the second integral

$$\frac{\nabla\mu_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\Phi_\delta}} = \sqrt{\Phi_\delta}\nabla\varphi_{\varepsilon,\delta}(\Phi_\delta) = \nabla V_{\varepsilon,\delta}(\Phi_\delta)$$

and we can prove that $\nabla V_{\varepsilon,\delta}(\Phi_\delta)$ weakly converges in $L^2((0,T)\times\Omega)$ towards ∇V_ε. By uniqueness of the limit in the sense of distribution we have $\sqrt{\Phi_{\varepsilon,\vartheta}}\nabla V_\varepsilon = \nabla\mu_\varepsilon(\Phi_{\varepsilon,\vartheta})$. Thus we can pass to the limit in the weak formulation (56) to obtain (22).

For the other diffusion term,

$$\int_0^T\int_\Omega \frac{\lambda_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)}}\sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)}\mathrm{div}\,u_\delta\mathrm{div}\,\zeta$$

we can prove the strong convergence in $L^2((0,T)\times\Omega)$ of the first part $\frac{\lambda_{\varepsilon,\delta}(\Phi_\delta)}{\sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)}}$. Indeed we have the bound (52) and the convergence a.e. since Φ_δ converges strongly to $\Phi_{\varepsilon,\vartheta}$ and $\Phi_{\varepsilon,\vartheta} < \Phi^*$ a.e.. On the other hand $\sqrt{\mu_{\varepsilon,\delta}(\Phi_\delta)}\mathrm{div}\,u_\delta$ converges weakly in $L^2((0,T)\times\Omega)$ towards $\sqrt{\mu_\varepsilon(\Phi_{\varepsilon,\vartheta})}\mathrm{div}\,u_{\varepsilon,\vartheta}$ (this is the previous point). We deduce then the convergence of the integral towards

$$\int_0^T\int_\Omega \frac{\lambda_\varepsilon(\Phi_{\varepsilon,\vartheta})}{\sqrt{\mu_\varepsilon(\Phi_{\varepsilon,\vartheta})}}\sqrt{\mu_\varepsilon(\Phi_{\varepsilon,\vartheta})}\mathrm{div}\,u_{\varepsilon,\vartheta}\mathrm{div}\,\zeta$$

Note finally that, at the limit $\delta = 0$, we have the relation

$$\pi_\varepsilon(\Phi_{\varepsilon,\vartheta}) = \left(\frac{\Phi_{\varepsilon,\vartheta}}{\Phi^*}\right)^\gamma \mu_\varepsilon^1(\Phi_{\varepsilon,\vartheta}), \qquad (57)$$

Remark on the limit passage $\vartheta \to 0$: Since at this stage we ensure that $\Phi_{\varepsilon,\vartheta}$ is bounded in $L^\infty((0,T)\times\Omega)$ we can deduce a control of $\nabla\Phi_{\varepsilon,\vartheta}$ which does not depend on ϑ. Indeed, thanks to the κ-entropy inequality (34) we have

$$\frac{\mu_\varepsilon'(\Phi_{\varepsilon,\vartheta})}{\sqrt{\Phi_{\varepsilon,\vartheta}}}\nabla\Phi_{\varepsilon,\vartheta} = \sqrt{\Phi_{\varepsilon,\vartheta}}\nabla\varphi_\varepsilon(\Phi_{\varepsilon,\vartheta}) \quad \text{bounded in} \quad L^\infty(0,T;L^2(\Omega)).$$

Then, since $\mu'_\varepsilon(\Phi_{\varepsilon,\vartheta}) \geq 1$ and $\Phi_{\varepsilon,\vartheta} \in L^\infty((0,T) \times \Omega)$, we get that

$$\nabla \Phi_{\varepsilon,\vartheta} \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)). \tag{58}$$

We can then pass to the limit $\vartheta \to 0$ in the equations to obtain system (10).

4 Proof of Theorem 2, recovering the two-phase system as $\varepsilon \to 0$

The aim of this section is to rigourously prove the limit passage from the suspension model (10) towards the two-phase system (24). What differs from the previous section is that at the limit volume fraction Φ can reach the constraint Φ^* on a set of positive measure. We expect then that the bounds on the diffusion terms will be more subtle because the singular terms involve the quantity

$$\frac{\varepsilon^{1+a}}{1 - \dfrac{\Phi_\varepsilon}{\Phi^*}}.$$

We see the competition between ε^{1+a} which tends to 0 and $1 - \dfrac{\Phi_\varepsilon}{\Phi^*}$ which can tend to 0 possibly faster than ε^{1+a}. In particular we do not have a uniform control on λ_ε and to pass to the limit in the corresponding diffusion term we will need to consider the renormalized continuity equation.

Estimates

Control of the volume fraction. From the previous step, we know that Φ_ε is in $L^\infty((0,T) \times \Omega)$ since it is bounded by Φ^*. As it has been explained for the limit passage $\vartheta \to 0$, we have in addition an uniform control of the gradient $\nabla \Phi_\varepsilon$ in $L^\infty(0,T;L^2(\Omega))$ since

$$|\nabla \Phi_\varepsilon| \leq \sqrt{\Phi^*} \frac{\sqrt{\mu'_\varepsilon(\Phi_\varepsilon)}}{\sqrt{\Phi_\varepsilon}} |\nabla \Phi_\varepsilon| \in L^\infty(0,T;L^2(\Omega)).$$

Integrability given by the drag term. As previously we ensure extra-integrability of $\sqrt{\Phi_\varepsilon} u_\varepsilon$ thanks to the the turbulent drag term present in the momentum equation

$$\Phi_\varepsilon^{1/3} u_\varepsilon \quad \text{is bounded in} \quad L^3((0,T) \times \Omega). \tag{59}$$

Control of the singular coefficients. Since the κ-entropy (21) gives

$$\frac{\nabla \mu_\varepsilon(\Phi_\varepsilon)}{\sqrt{\Phi_\varepsilon}} \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega))$$

and

$$r\mu_\varepsilon(\Phi_\varepsilon) \quad \text{is bounded in} \quad L^\infty(0,T;L^1(\Omega)),$$

since moreover $\Phi_\varepsilon \leq \Phi^*$, then

$$\mu_\varepsilon(\Phi_\varepsilon) \quad \text{is bounded in} \quad L^\infty(0,T;W^{1,2}(\Omega))$$

Thanks to the relationship between π_ε and μ_ε^1 we deduce that

$$\nabla \pi_\varepsilon(\Phi_\varepsilon) = \frac{\gamma \Phi_\varepsilon^{\gamma-1}}{(\Phi^*)^\gamma} \mu_\varepsilon^1(\Phi_\varepsilon) \nabla \Phi_\varepsilon + \left(\frac{\Phi_\varepsilon}{\Phi^*}\right)^\gamma \nabla \mu_\varepsilon^1(\Phi_\varepsilon)$$

$$\text{is bounded in} \quad L^\infty(0,T; L^q(\Omega)), \quad \forall\, q \in [1,2). \tag{60}$$

As for the previous step $\delta \to 0$, we need a control of the quantity $\dfrac{\mu_\varepsilon(\Phi_\varepsilon)}{\sqrt{\Phi_\varepsilon}}$. Let $\underline{\Phi} \in (0,1)$, on the set $\{\Phi_\varepsilon/\Phi^* \geq \underline{\Phi}\}$,

$$\frac{\mu_\varepsilon(\Phi_\varepsilon)}{\sqrt{\Phi_\varepsilon}} \leq \frac{\mu_\varepsilon(\Phi_\delta)}{\sqrt{\underline{\Phi}}\sqrt{\Phi^*}}$$

is bounded in $L^\infty(0,T; L^p(\Omega))$, $p < \infty$. On the other set $\{\Phi_\varepsilon/\Phi^* \leq \underline{\Phi}\}$, Φ_ε is far from Φ^* and then uniformly in ε

$$\frac{\mu_\varepsilon(\Phi_\varepsilon)}{\sqrt{\Phi_\varepsilon}} \leq \frac{\sqrt{\Phi_\varepsilon}}{\varepsilon} \left(\exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi_\varepsilon/\Phi^*}\right) - 1\right) \sqrt{\Phi_\varepsilon} \leq C\sqrt{\Phi_\varepsilon}$$

which is still bounded in $L^\infty(0,T; L^p(\Omega))$ for all $p \in [1,\infty)$. Therefore, in both cases

$$\frac{\mu_\varepsilon(\Phi_\varepsilon)}{\sqrt{\Phi_\varepsilon}} \quad \text{is bounded in} \quad L^\infty(0,T; L^p(\Omega)), \quad p \in [1,+\infty). \tag{61}$$

Passage to limit $\varepsilon \to 0$

With all these estimates we can now pass to the limit in the weak formulations of the mass and the momentum equations. The main convergence arguments remain the same as those presented in the previous section. We ensure with the Aubin-Lions-Simon lemma the strong convergence of the density Φ_ε towards Φ in $\mathcal{C}([0,T], L^p(\Omega))$ for all $p \in [1,+\infty)$ and the limit gradient of the density $\nabla\Phi$ is bounded in $L^\infty(0,T; L^2(\Omega))$. We can also obtain the strong convergence of the momentum $\Phi_\varepsilon u_\varepsilon$ towards m, define a limit velocity u equal to 0 on the set $\{\Phi = 0\}$ and such that $m = \Phi u$.

The procedure to pass to the limit in the turbulent drag term and the convective term is the same as previously and consists essentially in splitting the integral (for instance of the convective term)

$$\int_0^T \int_\Omega |\sqrt{\Phi_\varepsilon} u_\varepsilon - \sqrt{\Phi} u|^2$$

between small and large velocities. The Dominated Convergence Theorem gives the convergence to 0 for the small velocities whereas we need (59) to control the integral for the large velocities.

With the previous controls we get that

$$\begin{cases} \pi_\varepsilon(\Phi_\varepsilon) \longrightarrow \Pi & \text{weakly-* in} \quad L^\infty(0,T; W^{1,q}(\Omega)) \\ & \qquad\qquad\qquad \forall\, q \in [1,2) \\ \mu_\varepsilon(\Phi_\varepsilon) = \mu_\varepsilon^1(\Phi_\varepsilon) + \Phi_\varepsilon \longrightarrow \overline{\mu} + \Phi & \text{weakly-* in} \quad L^\infty(0,T; W^{1,2}(\Omega)) \end{cases}$$

Lemma 2 *At the limit $\varepsilon = 0$ we have*

$$(\Phi^* - \Phi)\overline{\mu} = 0 \tag{62}$$

$$(\Phi^* - \Phi)\Pi = 0 \tag{63}$$

and the equality

$$\overline{\mu} = \left(\frac{\Phi}{\Phi^*}\right)^{\gamma} \Pi = \Pi. \tag{64}$$

In particular,

$$\Pi \in L^{\infty}(0, T; W^{1,2}(\Omega)).$$

Proof. When $\varepsilon > 0$ we have

$$(1 - \Phi_\varepsilon/\Phi^*)\mu_\varepsilon^1(\Phi_\varepsilon) = \frac{\Phi_\varepsilon(1 - \Phi_\varepsilon/\Phi^*)}{\varepsilon}\left(\exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi_\varepsilon/\Phi^*}\right) - 1\right)$$

$$= \varepsilon^a \Phi_\varepsilon \frac{1 - \Phi_\varepsilon/\Phi^*}{\varepsilon^{1+a}}\left(\exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi_\varepsilon/\Phi^*}\right) - 1\right).$$

We need to consider separetely three differents subsets

$$\Omega^1 = \{\exists b < a, \text{ s.t. } \forall \varepsilon, \ 1 - \Phi_\varepsilon/\Phi^* \geq \varepsilon^b\}, \quad \Omega^2 = \{\exists c > 1 + a, \text{ s.t. } \forall \varepsilon, \ 1 - \Phi_\varepsilon/\Phi^* \leq \varepsilon^c\},$$

$$\Omega^3 = \{\forall \varepsilon, \ \varepsilon^{1+a} \leq 1 - \Phi_\varepsilon/\Phi^* \leq \varepsilon^a\}$$

- on Ω^1, this is the case where

$$\frac{1}{\varepsilon}\left(\exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi_\varepsilon/\Phi^*}\right) - 1\right) \to 0$$

  and for which we have directly the convergence of $(1 - \Phi_\varepsilon/\Phi^*)\mu_\varepsilon^1$ to 0.

- on Ω^2, the most singular case, we have

$$X = \frac{\varepsilon^{1+a}}{1 - \Phi_\varepsilon/\Phi^*} \longrightarrow +\infty$$

  and if $p > 0$ and X is large enough (or ε small enough), we can ensure that

$$\frac{1}{X} \leq (\exp(X) - 1)^p.$$

Using the fact that Φ_ε is bounded away from 0, we get

$$(1 - \Phi_\varepsilon/\Phi^*)\mu_\varepsilon^1 \leq \varepsilon^a \Phi_\varepsilon \left(\exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi_\varepsilon/\Phi^*}\right) - 1\right)^{1+p}$$

$$\leq \varepsilon^{a+1+p}\Phi_\varepsilon^{-p}\frac{\Phi_\varepsilon^{1+p}}{\varepsilon^{1+p}}\left(\exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi_\varepsilon/\Phi^*}\right) - 1\right)^{1+p}$$

$$= C\varepsilon^{a+1+p}\Phi_\varepsilon^{-p}(\mu_\varepsilon^1(\Phi_\varepsilon))^{1+p}$$

which tends to 0 since $\mu_\varepsilon^1(\Phi_\varepsilon)$ is bounded in $L^{\infty}(0, T; L^p(\Omega))$ for all $p \in [1, +\infty)$.

- on Ω^3, the intermediate case, we ensure that

$$\exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi_\varepsilon/\Phi^*}\right) - 1 \leq C$$

then, since we assumed that $a > 1$

$$(1 - \Phi_\varepsilon/\Phi^*)\mu_\varepsilon^1 \leq C\varepsilon^a \Phi_\varepsilon \frac{1 - \Phi_\varepsilon/\Phi^*}{\varepsilon^{1+a}}$$
$$\leq C\varepsilon^{a-1} \longrightarrow 0$$

Thus, in every case, $(1 - \Phi_\varepsilon/\Phi^*)\mu_\varepsilon^1$ converges strongly to 0 in $L^\infty(0,T;L^p(\Omega))$ for all $p \in [1, +\infty)$. With the same arguments, since $\pi_\varepsilon(\Phi_\varepsilon) = \left(\dfrac{\Phi_\varepsilon}{\Phi^*}\right)^\gamma \mu_\varepsilon^1(\Phi_\varepsilon)$ has the same divergence close to Φ^*, and since Φ_ε is in $L^\infty(0,T;L^p(\Omega))$, for all $p \in [1, +\infty)$, we ensure that

$$(1 - \Phi/\Phi^*)\,\Pi = 0.$$

By the strong convergence of Φ_ε and the weak convergences of $\mu_\varepsilon(\Phi_\varepsilon)$ and $\pi_\varepsilon(\Phi_\varepsilon)$ we get in addition

$$\Pi = \left(\frac{\Phi}{\Phi^*}\right)^\gamma \overline{\mu}.$$

Combined with two previous constraints, $\Pi = \overline{\mu} = 0$ on $\{\Phi < \Phi^*\}$, it gives finally

$$\Pi = \overline{\mu} \quad \text{a.e. on } (0,T) \times \Omega$$

and since $\overline{\mu}$ lies in $L^\infty(0,T;W^{1,2}(\Omega))$,

$$\Pi \in L^\infty(0,T;W^{1,2}(\Omega)). \qquad \square$$

Finally

$$\begin{cases} \pi_\varepsilon(\Phi_\varepsilon) \longrightarrow \Pi & \text{weakly-* in} \quad L^\infty(0,T;W^{1,q}(\Omega)) \\ & \qquad\qquad \forall\, q \in [1,2) \\ \mu_\varepsilon(\Phi_\varepsilon) = \mu_\varepsilon^1(\Phi_\varepsilon) + \Phi_\varepsilon \longrightarrow \Pi + \Phi & \text{weakly-* in} \quad L^\infty(0,T;W^{1,2}(\Omega)) \end{cases}$$

Concerning the diffusion term $\mu_\varepsilon(\Phi_\varepsilon)\,\mathrm{D}\,(u_\varepsilon)$, the weak formulation writes as

$$-\int_0^T \int_\Omega \sqrt{\Phi_\varepsilon}\, u_\varepsilon^j \left(\frac{\mu_\varepsilon(\Phi_\varepsilon)}{\sqrt{\Phi_\varepsilon}}\partial_{ij}^2\zeta^j + \frac{\mu_\varepsilon(\Phi_\varepsilon)}{\sqrt{\Phi_\varepsilon}}\partial_{ii}^2\zeta^j\right)$$
$$-\int_0^T \int_\Omega \sqrt{\Phi_\varepsilon}\, u_\varepsilon^i \left(\frac{\partial_j\mu_\varepsilon(\Phi_\varepsilon)}{\sqrt{\Phi_\varepsilon}}\partial_i\zeta^j + \frac{\partial_i\mu_\varepsilon(\Phi_\varepsilon)}{\sqrt{\Phi_\varepsilon}}\partial_i\zeta^j\right)$$

The first integral converges to

$$\int_0^T \int_\Omega \sqrt{\Phi}\, u^j \left(\frac{\Pi + \Phi}{\sqrt{\Phi}}\partial_{ij}^2\zeta^j + \frac{\Pi + \Phi}{\sqrt{\Phi}}\partial_{ii}^2\zeta^j\right)$$

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since $\dfrac{\mu_\varepsilon(\Phi_\varepsilon)}{\sqrt{\Phi_\varepsilon}}$ converges weakly-* in $L^\infty(0, T; L^p(\Omega))$ for all $p \in [1, +\infty)$. We recall that

$$\frac{\partial_j \mu_\varepsilon(\Phi_\varepsilon)}{\sqrt{\Phi_\varepsilon}} = \partial_j V_\varepsilon(\Phi_\varepsilon)$$

converges weakly in $L^2((0, T) \times \Omega)$ and by uniqueness of the limit in the sense of distribution

$$\sqrt{\Phi}\,\nabla V = \nabla\big(\Pi + \Phi\big).$$

Therefore the second integral converges to

$$\int_0^T \int_\Omega \sqrt{\Phi}\,u^i \left(\frac{\partial_j(\Pi + \Phi)}{\sqrt{\Phi}}\partial_i \zeta^j + \frac{\partial_i(\Pi + \Phi)}{\sqrt{\Phi}}\partial_i \zeta^j \right).$$

Transport equation relating Π and Λ.

Let us write the renormalized continuity equation on $\mu_\varepsilon^1(\Phi_\varepsilon)$,

$$\partial_t \mu_\varepsilon^1(\Phi_\varepsilon) + \operatorname{div}\left(\mu_\varepsilon^1(\Phi_\varepsilon) u_\varepsilon\right) + \frac{\lambda_\varepsilon(\Phi_\varepsilon)}{2}\operatorname{div} u_\varepsilon = 0 \tag{65}$$

or if we write the weak formulation

$$-\int_0^T \int_\Omega \mu_\varepsilon^1(\Phi_\varepsilon)\partial_t \xi - \int_0^T \int_\Omega \frac{\mu_\varepsilon^1(\Phi_\varepsilon)}{\sqrt{\Phi_\varepsilon}}\sqrt{\Phi_\varepsilon}u_\varepsilon \cdot \nabla\xi$$
$$-\frac{1}{2}\langle \lambda_\varepsilon(\Phi_\varepsilon)\operatorname{div} u_\varepsilon\,,\ \xi\rangle = \int_\Omega \mu_\varepsilon^1(\Phi_\varepsilon^0)\xi(0)$$

for $\xi \in \mathcal{D}([0, T) \times \Omega)$. Using the convergence already mentionned : $\mu_\varepsilon^1(\Phi_\varepsilon)$ converges weakly-* in $L^\infty(0, T; L^p(\Omega))$ for all $p \in [1, +\infty)$ towards Π, $\mu_\varepsilon^1(\Phi_\varepsilon)/\sqrt{\Phi_\varepsilon}$ converges weakly in $L^2((0, T) \times \Omega)$ towards $\Pi/\sqrt{\Phi}$ and $\sqrt{\Phi_\varepsilon}u_\varepsilon$ converges strongly to $\sqrt{\Phi}u$ in $L^2((0, T) \times \Omega)$, we deduce that $\lambda_\varepsilon(\Phi_\varepsilon)\operatorname{div} u_\varepsilon$ converges in the sense of distributions towards a weak limit denoted by Λ and belonging to $W^{-1,\infty}(0, T; L^p(\Omega)) \cap L^2(0, T; W^{-1,q}(\Omega))$ for all $p \in [1, +\infty)$, $q \in [1, 2)$. Moreover the equation (24c) is satisfied in the sense of distributions

$$\partial_t \Pi + \operatorname{div}(\Pi u) = -\frac{\Lambda}{2}$$

This completes the proof of Theorem 2.

5 Incompressible flows with pressure dependent viscosity

This last section is devoted to the proof of Theorem 3. Our first goal is to show that the limit continuity equation (24a) associated to the constraint $0 \leq \Phi \leq \Phi^*$ is compatible with the incompressibility condition $\operatorname{div} u = 0$ on the set $\{\Phi = \Phi^*\}$. Next we prove that the suspension model with initial density $\Phi_\varepsilon^0 = \Phi^*(1 - \varepsilon^a \Phi^*/\Pi^0)$ approximates thanks to Theorem 2 the fully incompressible system with pressure dependent viscosity.

We need to extend the compatibility lemma given by P.–L. LIONS and N. MASMOUDI in [24] to the case of degenerate viscosities.

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Proposition 2 (Compatibility relation) *Let (Φ, u) be such that*

$$\Phi \in L^p((0,T) \times \Omega) \quad \forall p \in [1, +\infty), \quad \nabla\Phi \in L^\infty(0,T; L^2(\Omega))$$

$$\sqrt{\Phi}\,\nabla u \in L^2((0,T) \times \Omega), \quad \Phi u \in L^\infty(0,T; L^q(\Omega)) \quad \text{with} \quad q > 1$$

satisfying the continuity equation

$$\partial_t \Phi + \text{div}\,(\Phi u) = 0 \quad in \quad (0,T) \times \Omega, \quad \Phi(0) = \Phi^0.$$

Then the following assertions are equivalent

1. *$\text{div}\, u = 0$ a.e. on $\{\Phi \geq \Phi^*\}$ and $0 \leq \Phi^0 \leq \Phi^*$.*

2. *$0 \leq \Phi \leq \Phi^*$*

Proof.

- $(1 \implies 2)$ As in [24], we set

$$\beta(r) = \begin{cases} 0 & \text{if} \quad r < 0 \\ r & \text{if} \quad 0 \leq r \leq \Phi^* \\ 1 & \text{if} \quad r > \Phi^* \end{cases}$$

and β_η a regular approximation of β such that $\beta_\eta(r) = \beta(r)$ on

$$(-\infty, -\eta) \cup (\eta, \Phi^* - \eta) \cup (\Phi^* + \eta, +\infty)$$

and such that

$$(\beta_\eta)'(\Phi)\Phi - \beta_\eta(\Phi) \leq C\sqrt{\Phi}.$$

Since $\nabla\Phi \in L^\infty(0,T; L^2(\Omega))$, we can multiply the continuity equation by $\beta_\eta'(\Phi)$ and obtain

$$\partial_t \beta_\eta(\Phi) + \text{div}\,(\beta_\eta(\Phi u)) + \big((\beta_\eta)'(\Phi)\Phi - \beta_\eta(\Phi)\big)\text{div}\, u = 0 \tag{66}$$

We have that $\beta_\eta(\Phi)$ converges pointwise and in $L^2((0,T) \times \Omega)$ to $\beta(\Phi)$. Moreover

$$\big((\beta_\eta)'(\Phi)\Phi - \beta_\eta(\Phi)\big)\text{div}\, u = \frac{(\beta_\eta)'(\Phi)\Phi - \beta_\eta(\Phi)}{\sqrt{\Phi}}\sqrt{\Phi}\,\text{div}\, u \quad \text{is bounded in} \quad L^2((0,T) \times \Omega)$$

and converges to $\mathbf{1}_{\{\Phi \geq \Phi^*\}}\text{div}\, u$. Then passing to the limit in (66) with respect to η and using the assumption $\text{div}\, u = 0$ on $\{\Phi \geq \Phi^*\}$, we get

$$\partial_t \beta(\Phi) + \text{div}\,(\beta(\Phi)u) = 0. \tag{67}$$

To conclude we set $d = \beta(\Phi) - \Phi$, regularizing the function $|d|$, we show as previously that $|d|$ satisfies

$$\begin{cases} \partial_t |d| + \text{div}\,(|d|u) = 0 \\ |d|(0) = 0 \end{cases}$$

Then integrating in space we get

$$\int_\Omega |d|(t) = \int_\Omega |d|(0) = 0$$

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and therefore
$$d(t) = 0 \quad \text{for all} \quad t$$
which means that $\beta(\Phi) = \Phi$ or
$$0 \leq \Phi \leq \Phi^*.$$

- (2 $\implies$ 1) Assuming that $0 \leq \Phi \leq \Phi^*$, equation (67) holds for $\beta(\Phi) = \left(\dfrac{\Phi}{\Phi^*}\right)^k$, for any integer k, since Φ^* is a constant

$$\partial_t \left(\frac{\Phi}{\Phi^*}\right)^k + \text{div}\left(\left(\frac{\Phi}{\Phi^*}\right)^k u\right) = (1-k)\left(\frac{\Phi}{\Phi^*}\right)^k \text{div}\, u \qquad (68)$$

On the left-hand side we have

$$\partial_t \left(\frac{\Phi}{\Phi^*}\right)^k \in W^{-1,\infty}(0,T;L^p(\Omega)) \quad , \quad \text{div}\left(\left(\frac{\Phi}{\Phi^*}\right)^k u\right) \in L^\infty(0,T,W^{-1,q}(\Omega))$$

for all $p \in [1,\infty)$ and for a $q \in (1,+\infty)$, which shows that the right–hand side of (68) is a bounded distribution. Then, if we let k go to $+\infty$ we obtain

$$\left(\frac{\Phi}{\Phi^*}\right)^k \text{div}\, u \longrightarrow 0 \quad \text{in} \quad \mathcal{D}'((0,T) \times \Omega).$$

On the other hand, $\left(\dfrac{\Phi}{\Phi^*}\right)^k \text{div}\, u$ converges pointwise to $\mathbf{1}_{\{\Phi=\Phi^*\}} \text{div}\, u$ and since

$$\left|\left(\frac{\Phi}{\Phi^*}\right)^k \text{div}\, u\right| \leq \frac{\Phi^{k-1/2}}{(\Phi^*)^k}\sqrt{\Phi}\,|\text{div}\, u| \quad \text{is bounded in} \quad L^2((0,T) \times \Omega)$$

we conclude by uniqueness of the limit in the sense of distribution that

$$\mathbf{1}_{\{\Phi=\Phi^*\}} \text{div}\, u = 0. \quad \square$$

5.1 Existence for the incompressible system with additional drag

Let us prove now the Theorem 3, we consider for that the approximate initial data

$$u_\varepsilon^0 \in L^2(\Omega), \quad \Phi_\varepsilon^0 = \Phi^*\left(1 - \varepsilon^a \frac{\Phi^*}{\Pi^0}\right) \qquad (69)$$

such that

$$\sqrt{\Phi_\varepsilon^0} u_\varepsilon^0 \to \sqrt{\Phi^*} u^0 \quad \text{in} \quad L^2(\Omega)$$

with ε small enough to ensure

$$1 - \varepsilon^a \frac{\Phi^*}{\text{ess inf}\, \Pi^0} > 0.$$

The approximate initial volume fraction obviously satisfies hypothesis (15), is positive thanks to the previous assumption and bounded uniformly with respect to ε in $W^{1,2}(\Omega)$. One can

also check condition (16), i.e. $|m_\varepsilon^0|^2/\Phi_\varepsilon^0 \in L^1(\Omega)$ since Φ_ε^0 is far from 0 and $m_\varepsilon^0 = \Phi_\varepsilon^0 u_\varepsilon^0$ is in $L^2(\Omega)$.

The approximate pressure $\pi_\varepsilon(\Phi_\varepsilon^0)$ converges a.e. to $\Pi^0 > 0$ since $a > 1$ and

$$
\begin{aligned}
\pi_\varepsilon(\Phi_\varepsilon^0) &= \left(\frac{\Phi_\varepsilon^0}{\Phi^*}\right)^\gamma \frac{\Phi_\varepsilon^0}{\varepsilon} \left(\exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi_\varepsilon^0/\Phi^*}\right) - 1\right) \\
&= \Phi^* \left(1 - \varepsilon^a \frac{\Phi^*}{\Pi^0}\right)^{\gamma+1} \frac{\exp(\varepsilon\Pi^0/\Phi^*) - 1}{\varepsilon} \\
&= \Phi^* \left(1 - \varepsilon^a(\gamma+1)\frac{\Phi^*}{\Pi^0} + o(\varepsilon^a)\right) \left(\frac{\Pi^0}{\Phi^*} + \varepsilon\frac{(\Pi^0)^2}{(\Phi^*)^2} + o(\varepsilon)\right) \\
&= \Pi^0 + \varepsilon\frac{(\Pi^0)^2}{\Phi^*} + o(\varepsilon)
\end{aligned}
$$

Let us check now that (17) and (18) are also satisfied. We have

$$
\begin{aligned}
\nabla\mu_\varepsilon(\Phi_\varepsilon^0) &= \left[\frac{1}{\varepsilon}\left(\exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi_\varepsilon^0/\Phi^*}\right) - 1\right) + \varepsilon^a\frac{\Phi_\varepsilon}{\Phi^*(1 - \Phi_\varepsilon^0/\Phi^*)^2}\right]\nabla\Phi_\varepsilon^0 \\
&= \varepsilon^a\left(\frac{\Phi^*}{\Pi^0}\right)^2\left[\frac{1}{\varepsilon}\left(\exp\left(\frac{\varepsilon\Pi^0}{\Phi^*}\right) - 1\right) + \varepsilon^{-a}\frac{\Phi_\varepsilon}{\Phi^*}\frac{(\Pi^0)^2}{(\Phi^*)^2}\right]\nabla\Pi^0
\end{aligned}
$$

then we can bound

$$
|\nabla\mu_\varepsilon(\Phi_\varepsilon^0)| \leq \left[\left(\frac{\Phi^*}{\Pi^0}\right)^2\left(\exp\left(\Pi^0/\Phi^*\right) - 1\right) + 1\right]|\nabla\Pi^0|
$$

Therefore, thanks to the control of Π^0 in $L^\infty(\Omega) \cap W^{1,2}(\Omega)$ we deduce that $\mu_\varepsilon(\Phi_\varepsilon^0)$ is controlled in $W^{1,2}(\Omega)$. Finally, since Φ_ε^0 is bounded by below, we check the condition (17)

$$
\|\frac{\nabla\mu_\varepsilon(\Phi_\varepsilon^0)}{\sqrt{\Phi_\varepsilon^0}}\|_{L^2} \leq C
$$

Concerning condition (18), we establish the following control

$$
\begin{aligned}
\Phi_\varepsilon^0 e_\varepsilon(\Phi_\varepsilon^0) &= \Phi_\varepsilon^0 \int_0^{\Phi_\varepsilon^0} \frac{\pi_\varepsilon(s)}{s^2}\,\mathrm{d}s \\
&= \frac{\Phi_\varepsilon^0}{\varepsilon(\Phi^*)^\gamma} \int_0^{\Phi_\varepsilon^0} s^{\gamma-1}\left(\exp\left(\frac{\varepsilon^{1+a}}{1 - s/\Phi^*}\right) - 1\right)\,\mathrm{d}s \\
&\leq \frac{\Phi_\varepsilon^0}{\varepsilon(\Phi^*)^\gamma} \int_0^{\Phi_\varepsilon^0} s^{\gamma-1}\left(\exp\left(\frac{\varepsilon^{1+a}}{1 - \Phi_\varepsilon^0/\Phi^*}\right) - 1\right)\,\mathrm{d}s \\
&\leq \frac{\Phi_\varepsilon^0}{\varepsilon(\Phi^*)^\gamma} \int_0^{\Phi_\varepsilon^0} s^{\gamma-1}\left(\exp\left(\frac{\varepsilon\Pi^0}{\Phi^*}\right) - 1\right)\,\mathrm{d}s \\
&\leq \frac{\exp\left(\frac{\varepsilon\Pi^0}{\Phi^*}\right) - 1}{\varepsilon\gamma}\frac{(\Phi_\varepsilon^0)^{\gamma+1}}{(\Phi^*)^\gamma} \\
&\leq \frac{\exp\left(\Pi^0/\Phi^*\right) - 1}{\gamma}\Phi_\varepsilon^0
\end{aligned}
$$

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The last quantity is then bounded in $L^1((0,T) \times \Omega)$ as desired since Π^0 is in $L^\infty(\Omega)$ and Φ_ε^0 is in $L^1(\Omega)$. This proves the condition (18).

If we consider now the solution $(\Phi_\varepsilon, u_\varepsilon, \pi_\varepsilon(\Phi_\varepsilon))$ of (10a)–(10b) with initial data $(\Phi_\varepsilon^0, m_\varepsilon^0)$, by the conservation of the mass we have

$$\Phi^* \left(1 - \varepsilon^a \frac{\Phi^*}{\operatorname{ess\,inf} \Pi^0} \right) |\Omega| \leq \int_\Omega \Phi_\varepsilon^0 = \int_\Omega \Phi_\varepsilon < \Phi^* |\Omega| \tag{70}$$

and therefore

$$\int_\Omega \Phi_\varepsilon \to \Phi^* |\Omega|. \tag{71}$$

Theorem 2 ensures the a.e. convergence of Φ_ε towards a limit Φ. Moreover, Φ_ε satisfies the constraint

$$0 \leq \Phi_\varepsilon \leq \Phi^* \tag{72}$$

Then, necessarily by conditions (71)–(72) we have $\Phi = \Phi^*$ a.e. on $(0,T) \times \Omega$ which proves with Proposition 2 the existence of a global weak solution (u, Π, Λ) to

$$\begin{cases} \operatorname{div} u = 0 & (73a) \\[2ex] -\int_0^T \int_\Omega \Pi \partial_t \xi - \int_0^T \int_\Omega \Pi u \cdot \nabla \xi + \langle \frac{\Lambda}{2}, \xi \rangle = \int_\Omega \Pi^0 \xi(0) & (73b) \\[2ex] -\int_0^T \int_\Omega u \cdot \partial_t \zeta - \int_0^T \int_\Omega (u \otimes u) : \nabla \zeta - \int_0^T \int_\Omega \frac{\Pi}{\Phi^*} \operatorname{div} \zeta + r \int_0^T \int_\Omega |u| u \\[1ex] \quad + \langle \frac{\Lambda}{\Phi^*}, \operatorname{div} \zeta \rangle + 2 \langle (\Pi/\Phi^* + 1) \operatorname{D}(u), \nabla \zeta \rangle = \int_\Omega u^0 \cdot \zeta(0), & (73c) \end{cases}$$

Acknowledgments

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